uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.4: Z-transform

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

4 Z-transform

Problem 4.1 Find the Z-transform of the following functions (with sampling period T and $\alpha \in \mathbb{C}$):

$$\textbf{1. } f(x) = 1\,, \qquad \textbf{2. } f(x) = x\,,$$
$$\textbf{3. } f(x) = x^2\,, \qquad \textbf{4. } f(x) = x^3\,,$$
$$\textbf{5. } f(x) = e^{\alpha x}\,, \qquad \textbf{6. } f(x) = \cos(\alpha x)\,,$$

Problem 4.2 Find the Z-transform of the following sequences (with $a, b, \alpha, \beta \in \mathbb{C}$):

$$\textbf{1. } f_n = 1\,, \qquad \textbf{2. } f_n = n\,,$$
$$\textbf{3. } f_n = n^2\,, \qquad \textbf{4. } f_n = n^3\,,$$
$$\textbf{5. } f_n = e^{\alpha n+\beta}\,, \qquad \textbf{6. } f_n = a^n\,,$$
$$\textbf{7. } f_n = \cos(\alpha n + \beta)\,, \qquad \textbf{8. } f_n = \sin(\alpha n + \beta)\,,$$
$$\textbf{9. } f_n = n\,a^n\,, \qquad \textbf{10. } f_n = n^2 a^n\,,$$
$$\textbf{11. } f_n = n^3 a^n\,, \qquad \textbf{12. } f_n = a^{n-1}\,,\ \text{if}\ \ n \geq 1\,,\ f_0 = 0\,,$$
$$\textbf{13. } f_n = \frac{a^n - b^n}{a-b}\,,\ a \neq b\,, \qquad \textbf{14. } f_n = \frac{a^{n+1} - b^{n+1}}{a-b}\,,\ a \neq b\,.$$

Hint: Some Z-transforms can be deduced from the previous problem.

Problem 4.3 Prove Theorem 4.3.

Problem 4.4 Prove Theorem 4.5.

Problem 4.5 Prove Theorem 4.6.

Problem 4.6 Find the Z-transform of the sequence f_n by using the items (1) and (2) in Theorem 2. Deduce the general term of the sequence f_n, by using some previous problem.

1. $f_{n+1} - 2f_n = 5$, with $f_0 = -4\,,$
2. $f_{n+2} - 2f_{n+1} + f_n = 0\,,$ with $f_0 = 1\,,\ f_1 = 4\,,$
3. $f_{n+2} = f_{n+1} + f_n\,,$ with $f_0 = 1\,,\ f_1 = 1\,,$
4. $f_{n+3} - 6f_{n+2} + 12f_{n+1} - 8f_n = 0\,,$ with $f_0 = 0\,,\ f_1 = 2\,,\ f_2 = 16$
5. $f_{n+2} - f_{n+1} - 2f_n = 0\,,$ with $f_0 = 0\,,\ f_1 = 1\,,$
6. $f_{n+1} + 3f_n = n\,,$ with $f_0 = -2/3\,,$
7. $f_{n+2} - 3f_{n+1} + 2f_n = -1\,,$ with $f_0 = -3/4\,,\ f_1 = 1/2\,,$
8. $4f_{n+2} + 4f_{n+1} + f_n = d_n\,,$ with $d_0 = 1\,,\ d_n = 0\ \ (n \geq 1),\ f_0 = 0\,,\ f_1 = 1/4\,.$

Hint: Apply the Z-transform to both sides of each equation and use the formula relating $(Zf_{n+k})(z)$ with $(Zf_n)(z)$.

A *digital filter* is an operator which maps a sequence f_n (called *input*) on other sequence g_n (called *output*) by using the formula

$$g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = \sum_{k=0}^{M} \alpha_k f_{n-k}\,, \qquad n \in \mathbb{N}\,,$$

where $\alpha_k, \beta_k \in \mathbb{C}$ and $M, N \in \mathbb{N}$ are fixed constants, and $f_m = g_m = 0$ if $m < 0$.

Problem 4.7 Prove that if f_n and g_n are Z-transformable, then

$$(Zg_n)(z) = H(z)\,(Zf_n)(z)\,, \qquad \text{with} \qquad H(z) = \frac{\sum_{k=0}^{M} \alpha_k z^{-k}}{1 + \sum_{j=1}^{N} \beta_j z^{-j}}\,,$$

for z with large enough modulus. The function $H(z)$ is called digital filter *transfer function.*

Problem 4.8 The sequence h_n whose Z-transform is the transfer function H is called *shock response.* Prove that the output g_n is the convolution of h_n and the input f_n.

Problem 4.9 Consider the following digital filters with their inputs. In each case calculate the transfer function and the output:

1. $g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = \sum_{k=0}^{M} \alpha_k f_{n-k}$, with $f_n = 0$.

2. $g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = f_n + \sum_{j=1}^{N} \beta_j f_{n-j}$, with any f_n.

3. $g_n + 3g_{n-1} + 2g_{n-2} = f_n$, with $f_0 = 1$, $f_n = 0$ if $n \geq 1$.

4. $g_n - g_{n-1} = f_n$, with $f_n = 1$.

5. $g_n + g_{n-1} = 3f_n + f_{n-1}$, with $f_n = 2n$.

Z-TRANSFORMS TABLE

$(a, b, \alpha, \beta \in \mathbb{C})$

$$Z[1](z) = \frac{z}{z-1}\,, \qquad Z[a^n](z) = \frac{z}{z-a}\,,$$

$$Z[n](z) = \frac{z}{(z-1)^2}\,, \qquad Z[n\,a^n](z) = \frac{az}{(z-a)^2}\,,$$

$$Z[n^2](z) = \frac{z(z+1)}{(z-1)^3}\,, \qquad Z[n^2\,a^n](z) = \frac{az(z+a)}{(z-a)^3}\,,$$

$$Z[n^3](z) = \frac{z(z^2+4z+1)}{(z-1)^4}\,, \qquad Z[n^3\,a^n](z) = \frac{az(z^2+4az+a^2)}{(z-a)^4}\,,$$

$$Z[\cos(\alpha n + \beta)](z) = \frac{z^2\cos\beta - z\cos(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1}\,, \qquad Z[\sin(\alpha n + \beta)](z) = \frac{z^2\sin\beta + z\sin(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1}\,,$$

$$Z\Big[\frac{a^n - b^n}{a-b}\Big](z) = \frac{z}{(z-a)(z-b)} \quad (a \neq b), \qquad Z\Big[\frac{a^{n+1} - b^{n+1}}{a-b}\Big](z) = \frac{z^2}{(z-a)(z-b)} \quad (a \neq b),$$

$$Z[f_n](z) = \frac{1}{z-a}\,, \qquad \text{if } f_n = a^{n-1}\,,\ n \geq 1\,,\ f_0 = 0\,.$$