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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.1: Fourier series

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

1 Fourier series

Problem 1.1 Given $k \geq 1$ and $a \in \mathbb{R}$, compute the Fourier series of the following functions defined on $[-L, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = 1\,, \qquad &&\mathbf{2.}\ f(x) = \cos\frac{k\pi x}{L}\,, \\ &\mathbf{3.}\ f(x) = x\,, \qquad &&\mathbf{4.}\ f(x) = x^2 \\ &\mathbf{5.}\ f(x) = e^{ax}, \qquad &&\mathbf{6.}\ f(x) = x^{-2}, \end{aligned}$$

$$\mathbf{7.}\ f(x) = \frac{R^2 - r^2}{R^2 - 2Rr\cos x + r^2}\,, \quad \text{with } L = \pi \text{ and } 0 < r < R\,.$$

Solution: 1) $a_0 = 2$, $a_n = b_n = 0$ for every $n \geq 1$. 2) $a_k = 1$, $a_n = 0$ for every $n \neq k$, $b_n = 0$ for every $n \geq 1$. 3) $a_n = 0$ for $n \geq 0$, $b_n = (-1)^{n+1}\frac{2L}{n\pi}$. 4) $b_n = 0$ for $n \geq 1$, $a_0 = \frac{2}{3}L^2$, $a_n = (-1)^n\frac{4L^2}{n^2\pi^2}$ for $n \geq 1$. 5) $a_n = (-1)^n\frac{2aL\sinh aL}{a^2L^2+n^2\pi^2}$, $b_n = (-1)^{n+1}\frac{2\pi n\sinh aL}{a^2L^2+n^2\pi^2}$. 6) $f(x)$ is not integrable on $[-L, L]$; hence, the Fourier coefficients do not exist and so, f does not have Fourier series. 7) $b_n = 0$; if γ is the unit circumference positively oriented, with $z = e^{ix}$, $dz = ie^{ix}dx$, $x \in [-\pi, \pi]$, residue theorem gives

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos x + r^2}\cos nx\,dx = \operatorname{Re}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos x + r^2}\,e^{inx}\,dx\Big) \\ &= \operatorname{Re}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{R^2 - r^2}{R^2 - Rr(e^{ix} + e^{-ix}) + r^2}\,e^{inx}\,\frac{ie^{ix}dx}{ie^{ix}}\Big) \\ &= \operatorname{Re}\Big(\frac{1}{\pi}\int_{\gamma}\frac{R^2 - r^2}{R^2 - Rr(z + z^{-1}) + r^2}\,z^n\,\frac{dz}{iz}\Big) \\ &= \operatorname{Re}\Big(\frac{1}{\pi i}\int_{\gamma}\frac{R^2 - r^2}{(R^2 + r^2)z - Rr(z^2 + 1)}\,z^n\,dz\Big) \\ &= \operatorname{Re}\Big(\frac{-1}{\pi iRr}\int_{\gamma}\frac{(R^2 - r^2)z^n}{z^2 - (R/r + r/R)z + 1}\,dz\Big) \\ &= \operatorname{Re}\Big(\frac{-1}{\pi iRr}\int_{\gamma}\frac{(R^2 - r^2)z^n}{(z - R/r)(z - r/R)}\,dz\Big) = \operatorname{Re}\Big(\frac{-2}{Rr}\operatorname{Res}\Big(\frac{(R^2 - r^2)z^n}{(z - R/r)(z - r/R)}\,,\frac{r}{R}\Big)\Big) \\ &= \operatorname{Re}\Big(\frac{-2}{Rr}\lim_{z\to r/R}\frac{(R^2 - r^2)z^n}{z - R/r}\Big) = \operatorname{Re}\Big(\frac{2}{Rr}\frac{(r^2 - R^2)(r/R)^n}{r/R - R/r}\Big) = 2\,\frac{r^n}{R^n}\,. \end{aligned}$$

Problem 1.2 Given $a \in \mathbb{R}$, compute the Fourier cosine series of the following functions defined on $[0, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = 1\,, \qquad &&\mathbf{2.}\ f(x) = e^{ax}, \\ &\mathbf{3.}\ f(x) = x\,, \qquad &&\mathbf{4.}\ f(x) = x^2. \end{aligned}$$

Solution: By definición, we have $b_n = 0$ for $n \geq 1$ for every function.
1) As the even extension of $f(x)$ is $f(x) = 1$ on the interval $[-L, L]$, (SF1) gives $a_0 = 2$, $a_n = 0$ for every $n \geq 1$.

2) By using formula (I6), we obtain

$$a_n = \frac{2}{L}\int_0^L e^{ax}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{1}{a^2+(n\pi/L)^2}\,e^{ax}\Big(\frac{n\pi}{L}\sin\frac{n\pi x}{L} + a\cos\frac{n\pi x}{L}\Big)\Big]_{x=0}^{x=L}$$
$$= \frac{2aL}{a^2L^2+n^2\pi^2}\big(e^{aL}\cos n\pi - 1\big) = \frac{2aL}{a^2L^2+n^2\pi^2}\big((-1)^n e^{aL} - 1\big)\,.$$

3) Formula (I4) gives

$$a_0 = \frac{2}{L}\int_0^L x\,dx = L\,,$$
$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\,x\sin\frac{n\pi x}{L}\Big]_{x=0}^{x=L} - \frac{2}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx$$
$$= 0 + \frac{2L}{n^2\pi^2}\cos\frac{n\pi x}{L}\Big]_{x=0}^{x=L} = \frac{2L}{n^2\pi^2}\big(\cos n\pi - 1\big) = \big((-1)^n - 1\big)\frac{2L}{n^2\pi^2}\,.$$

4) Since the even extension of $f(x)$ is $f(x) = x^2$ on $[-L, L]$, (SF5) gives $a_0 = 2L^2/3$, $a_n = (-1)^n 4L^2/(n\pi)^2$ for every $n \geq 1$.

Problem 1.3 Given $a \in \mathbb{R}$, compute the Fourier sine series of the following functions defined on $[0, L]$:

1. $f(x) = 1\,,$ **2.** $f(x) = e^{ax},$

3. $f(x) = x\,,$ **4.** $f(x) = x^2.$

Solution: By definición, we have $a_n = 0$ for $n \geq 0$ for every function.
1) We have for $n \geq 1$

$$b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{-2}{n\pi}\cos\frac{n\pi x}{L}\Big]_{x=0}^{x=L} = \frac{-2}{n\pi}\big(\cos n\pi - 1\big) = \big(1 + (-1)^{n+1}\big)\,\frac{2}{n\pi}\,.$$

2) (I7) gives for $n \geq 1$

$$b_n = \frac{2}{L}\int_0^L e^{ax}\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\,\frac{1}{a^2+(n\pi/L)^2}\,e^{ax}\Big(a\sin\frac{n\pi x}{L} - \frac{n\pi}{L}\cos\frac{n\pi x}{L}\Big)\Big]_{x=0}^{x=L}$$
$$= \frac{2\pi n}{a^2L^2+n^2\pi^2}\big(-e^{aL}\cos n\pi + 1\big) = \frac{2\pi n}{a^2L^2+n^2\pi^2}\big(1 + (-1)^{n+1}e^{aL}\big)\,.$$

3) As the odd extension of $f(x)$ is $f(x) = x$ on $[-L, L]$, (SF4) gives $b_n = (-1)^{n+1}2L/(n\pi)$ for $n \geq 1$.
4) Formulas (I5) and (I4) give for $n \geq 1$,

$$b_n = \frac{2}{L}\int_0^L x^2\sin\frac{n\pi x}{L}\,dx = \frac{-2}{n\pi}\,x^2\cos\frac{n\pi x}{L}\Big]_{x=0}^{x=L} + \frac{4}{n\pi}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$
$$= \frac{-2}{n\pi}\,L^2\cos n\pi + \frac{4}{n\pi}\Big(\frac{L}{n\pi}\,x\sin\frac{n\pi x}{L}\Big]_{x=0}^{x=L} - \frac{L}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx\Big)$$
$$= (-1)^{n+1}\frac{2L^2}{n\pi} + 0 + \frac{4L^2}{n^3\pi^3}\cos\frac{n\pi x}{L}\Big]_{x=0}^{x=L} = (-1)^{n+1}\frac{2L^2}{n\pi} - \big(1 + (-1)^{n+1}\big)\frac{4L^2}{n^3\pi^3}\,.$$

Problem 1.4 Given $a \in \mathbb{R}$, compute the Fourier series of the following functions defined on $[-L, L]$:

1. $f(x) = |x|\,,$ **2.** $f(x) = x|x|\,,$

3. $f(x) = e^{a|x|},$ **4.** $f(x) = \dfrac{x}{|x|}\,.$

Hint: Use the two previous problems.

Solution: 1) It is the cosine series of the function x (see Exercise 1.2). 2) It is the sine series of the function x^2 (see Exercise 1.3). 3) It is the cosine series of the function e^{ax} (see Exercise 1.2). 4) It is the sine series of the function 1 (see Exercise 1.3).

Problem 1.5 Prove the equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \,.$$

Hint: Apply (FSe) to the function in (FS4).

Solution: By applying (SFe) to the function in (SF4), since $a_n = 0$ for every $n \geq 0$, we have

$$\frac{2L^2}{3} = \frac{1}{L} \int_{-L}^{L} x^2 \, dx = \sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} \frac{4L^2}{n^2 \pi^2} \,,$$

and so,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \,.$$

Problem 1.6 Prove that if f is continuous, f' is piecewise continuous on $[-L, L]$ and $f(-L) = f(L)$, then

$$a_0(f') = 0 \,, \quad a_n(f') = \frac{n\pi}{L} \, b_n(f) \quad \text{and} \quad b_n(f') = \frac{-n\pi}{L} \, a_n(f) \ (n \geq 1) \,.$$

Hint: Apply integration by parts.

Problem 1.7 Prove that if f and f' are continuous, f'' is piecewise continuous on $[-L, L]$, $f(-L) = f(L)$, and $f'(-L) = f'(L)$, then

$$a_0(f'') = 0 \,, \quad a_n(f'') = -\Big(\frac{n\pi}{L}\Big)^2 a_n(f) \quad \text{and} \quad b_n(f'') = -\Big(\frac{n\pi}{L}\Big)^2 b_n(f) \ (n \geq 1) \,.$$

Hint: Use the previous exercise twice.

Problem 1.8 (1) Find a formula for the complex Fourier coefficients c_n in terms of the (real) Fourier coefficients a_n and b_n.
Hint: Use the identities $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.
(2) Find formulas for the (real) Fourier coefficients a_n and b_n, in terms of the complex Fourier coefficients c_n.
Hint: Use the previous item.

Problem 1.9 At which points x of the interval $[-L, L]$ does the Fourier series of f converge to $f(x)$?

$$\textbf{1.}\ f(x) = x \,, \qquad \textbf{2.}\ f(x) = x^2 .$$

Solution: Since $f(x)$ is piecewise C^1 on $[-L, L]$ in each case, its Fourier series converges to $f(x)$ if x is a continuity point of the periodic extension of f; if x is not a continuity point of the periodic extension of f, then its Fourier series converges to the mean value of the one-sided limits of the periodic extension of f at x. Hence:
1) Since the periodic extension of f is continuous on $(-L, L)$ and discontinuous at $x = -L$ and $x = L$, since $f(L) = L \neq -L = f(-L)$, the Fourier series converges to $f(x)$ for every $x \in (-L, L)$,

and if does not converge to $f(x)$ at $x = -L$ and $x = L$ (it converges to $(L - L)/2 = 0$ at both points).
2) Since $f(L) = L^2 = f(-L)$, the periodic extension of f is continuous on $\mathbb{R}$ and so, the Fourier series converges to $f(x)$ for every $x \in \mathbb{R}$.

Problem 1.10 Solve the heat equation on a rod with zero temperature at finite ends ($0 \leq x \leq L, t \geq 0$):

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } 0 < x < L\,,\, t > 0\,, \\ u(0,t) \;=\; 0\,, \quad u(L,t) \;=\; 0\,, & \text{if } t > 0\,, \\ u(x,0) \;= \sin\frac{2\pi x}{L}\,, & \text{if } 0 < x < L\,. \end{cases}$$

Solution: $u(x,t) = \sin\frac{2\pi x}{L}\, e^{-k(2\pi/L)^2 t}$.

Problem 1.11 Solve the heat equation on a rod with isolated endpoints ($0 \leq x \leq L, t \geq 0$):

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } 0 < x < L\,,\, t > 0\,, \\ \frac{\partial}{\partial t}u(0,t) \;=\; 0\,, \quad \frac{\partial}{\partial t}u(L,t) \;=\; 0\,, & \text{if } t > 0\,, \\ u(x,0) \;=\; f(x)\,, & \text{if } 0 < x < L\,. \end{cases}$$

Solution: $u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L}\, e^{-k(n\pi/L)^2 t}$, with $c_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$ for $n \geq 1$.

Problem 1.12 Solve the Laplace equation on the square $\{(x,y) \in \mathbb{R}^2 :\ 0<x<\pi,\ 0<y<\pi\}$:

$$\begin{cases} \Delta u = \frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0\,, \\ u(0,y) = u(\pi,y) = 0\,, \\ u(x,0) = 2\sin 3x\,,\ \ u(x,\pi) = 0\,. \end{cases}$$

Hint: Check that the function $v(y) = c_1 e^{\mu y} + c_2 e^{-\mu y} = k_1 \cosh\mu(\pi - y) + k_2 \sinh\mu(\pi - y)$ is solution of the differential equation $v''(y) - \mu^2 v(y) = 0$.
Solution: $u(x,y) = \sum_{n=1}^{\infty} a_n \sin nx \sinh n(\pi - y)$; since $u(x,0) = 2\sin 3x = \sum_{n=1}^{\infty} a_n \sin nx \sinh n\pi$, we have $a_3 = 2/\sinh 3\pi$ and $a_n = 0\ \forall n \neq 3$, and so, $u(x,y) = 2\sin 3x \sinh 3(\pi - y)/\sinh 3\pi$.