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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.1: Complex numbers

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1.1. COMPLEX NUMBERS

1.1. Compute the real and imaginary parts of

$$a)\ w = \frac{1}{i} + \frac{1}{1+i}\,, \qquad b)\ w = \frac{1}{(3+2i)^2}\,, \qquad c)\ w = \frac{z+1}{z^2+5} \quad \text{if } z = x+iy,\ x,y \in \mathbf{R}\,.$$

Solutions: *a*) $\operatorname{Re} w = 1/2$, $\operatorname{Im} w = -3/2$. *b*) $\operatorname{Re} w = 5/169$, $\operatorname{Im} w = -12/169$.

$$c) \qquad \operatorname{Re} w = \frac{(x+1)(x^2-y^2+5)+2xy^2}{(x^2-y^2+5)^2+4x^2y^2}\,, \qquad \operatorname{Im} w = \frac{-x^2y-y^3+5y-2xy}{(x^2-y^2+5)^2+4x^2y^2}\,.$$

1.2. Find every complex solution of each equation:

$$a)\ z^2 = 3-4i\,, \qquad b)\ (z+1)^4 + i = 0\,.$$

Hints: a) $z = \sqrt{3-4i}\,$. b) $z = (-i)^{1/4}$. Recall that $(re^{i\theta})^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$ $(k \in \mathbf{Z})$.

1.3. Compute:

$$a)\ (1+i)^4 \qquad b)\ (-i)^{-1} \qquad c)\ \sqrt{1+i} \qquad d)\ \sqrt{1+\sqrt{i}} \qquad e)\sqrt{\sqrt{-i}}$$

$$f)\ (1+i)^n + (1-i)^n \qquad g)\ \sqrt{1-i\sqrt{3}} \qquad h)\ i^{1/4} \qquad i)\ (-i)^{1/4}\,.$$

Hint: Recall that $(re^{i\theta})^{1/n} = r^{1/n}e^{i(\theta+2k\pi)/n}$ $(k \in \mathbf{Z})$.

1.4. Prove that the roots of the equation $z^n = a$, with $a \neq 0$, are the vertices of a regular polygon.
Solutions: If $a = re^{i\theta}$, then the roots are $z_k = r^{1/n}e^{i\theta/n}e^{i2\pi k/n}$ $(k \in \mathbf{Z})$, i.e., $z_{k+1} = z_k e^{i2\pi/n}$.

1.5. Show that every complex numbers z, w satisfy

$$|z-w|^2 + |z+w|^2 = 2\left(|z|^2 + |w|^2\right).$$

Hint: $|a|^2 = a\overline{a}$ for every $a \in \mathbf{C}$.

1.6. Compute the moduli of the following complex numbers:

$$a)\ z = i(2+3i)(5-2i)(-2-i)^{-1}, \qquad b)\ w = \frac{(2-3i)^2}{(8+6i)^2}\,.$$

Hint: $|ab| = |a||b|\,,\ |a/b| = |a|/|b|$ and $|a^n| = |a|^n$.

1.7. Let w be a n-th root of the unity, i.e., $w^n = 1$. Show that if $w \neq 1$, then:

$$a)\ 1 + w + w^2 + \cdots + w^{n-1} = 0\,, \qquad b)\ 1 + 2w + 3w^2 + \cdots + nw^{n-1} = \frac{n}{w-1}\,.$$

What are the values if $w = 1$?
Hints: *a*) It is the sum of a geometric progression. *b*) It is the sum of an arithmetic-geometric progression.

1.8. a) Prove Lagrange's trigonometric identities:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2}\left(1 + \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}\right),$$

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin((n+1)\theta/2)\,\sin(n\theta/2)}{\sin(\theta/2)}\,,$$

where θ is an angle such that $\sin(\theta/2) \neq 0$.

Hint: Use the formula for the sum of a geometric progression

$$1 + w + w^2 + \cdots + w^n = \frac{w^{n+1} - 1}{w - 1}\,, \qquad \text{with}\, w = e^{i\theta}\,.$$

b) Compute the sums:

$$\cos(\alpha + \theta) + \cos(\alpha + 2\theta) + \cdots + \cos(\alpha + n\theta)\,,$$
$$\sin(\alpha + \theta) + \sin(\alpha + 2\theta) + \cdots + \sin(\alpha + n\theta)\,.$$

Hint: Use the formulas in the previous item.

1.9. Describe the set of complex numbers satisfying:

$$a)\ \ \mathrm{Im}\,(z + 5) = 0\,, \qquad b)\ \ \mathrm{Re}\,(z^2 + 5) = 0\,.$$

Solutions: *a)* If $z = x + iy$, then $y = 0$, the real line. *b)* If $z = x + iy$, then $x^2 - y^2 = -5$, a hyperbola.

1.10 Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that

$$\Big|\sum_{j=1}^{n} z_j\Big| \leq \sum_{j=1}^{n} |z_j|\,.$$

Hint: The case $n = 2$ is the triangle inequality. Use induction for the general case.

1.11. If $|z| = 1$, prove that we have, for every complex numbers a, b,

$$\left|\frac{az + b}{\bar{b}z + \bar{a}}\right| = 1\,.$$

Hint: Use that $|az + b| = |\overline{z}(az + b)|$ (since $|\overline{z}| = |z| = 1$) and $|\overline{w}| = |w|$ for every $w \in \mathbf{C}$.

1.12. Consider the complex number $z = (1 + i\sqrt{3}\,)/(1 - i\sqrt{3}\,)$.

a) Compute its binomial and exponential forms.
b) Prove that $z^4 = z$.
c) Compute the other fourth roots of z.
d) Compute every complex number which is fourth root of itself.

Solutions: a) $z = \frac{2\,e^{i\pi/3}}{2\,e^{-i\pi/3}} = e^{2i\pi/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$. b) $z^4 = (e^{2i\pi/3})^4 = e^{8i\pi/3} = e^{2i\pi/3}e^{2i\pi} = e^{2i\pi/3} = z$. c) $e^{i\pi/6}$, $e^{7i\pi/6}$, $e^{5i\pi/3}$. d) $w^4 = w$, i.e., $w(w^3 - 1) = 0$. Hence, $w = 0$ or $w = \sqrt[3]{1} = \sqrt[3]{e^{2ki\pi}} = e^{2ki\pi/3}$, with $k = 0, 1, 2$.

1.13. Find the locus of the points z of the complex plane verifying $|z - z_1| = |z - z_2|$, if z_1 and z_2 are two complex numbers.

Solution: If $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the condition $|z - z_1| = |z - z_2|$ is equivalent to $2(x_2 - x_1)x + 2(y_2 - y_1)y + |z_1|^2 - |z_2|^2 = 0$, which is the equation of the straight line perpendicular to the segment joining z_1 and z_2, and passing through the midpoint of that segment (the segment's bisector).

1.14. Let z_1, z_2, z_3 be three vertices of a parallelogram. Find its fourth vertex z_4 opposite to z_2.

Hint: Use the geometric interpretation of the sum of complex numbers.

1.15. Let z_1, z_2 be two vertices of an equilateral triangle. Find its third vertex z_3.

Hint: Use the geometric interpretation of the product of complex numbers.