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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.8: Residue theorem

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1.8. RESIDUE THEOREM

8.1. Compute the following integrals by using the residue theorem:

a) $\displaystyle\int_\gamma \frac{dz}{z^2-1}$, with $\gamma(t) = 2\,e^{it}$, $t \in [0, 2\pi]$.

b) $\displaystyle\int_\gamma \frac{dz}{z^4+1}$, where γ is the boundary of the domain $\{x+iy : x^2+y^2<4,\, x>0\}$.

c) $\displaystyle\int_\gamma \frac{dz}{z^4+1}$, where γ is the circumference centered at 0 with radius 2.

d) $\displaystyle\int_\gamma \frac{1+z}{1-\cos z}\,dz$, where γ is the circumference centered at 0 with radius 7.

e) $\displaystyle\int_\gamma \frac{2+3\,\sin \pi z}{z(z-1)^2}\,dz$, where γ is the square with vertices $3+3i$, $3-3i$, $-3+3i$ and $-3-3i$.

f) $\displaystyle\int_\gamma e^{-1/z}\sin(1/z)\,dz$, where γ is the unit circumference.

g) $\displaystyle\int_\gamma \frac{dz}{(z^2+1)(z-1)^2}$, where γ is the circumference centered at $1+i$ with radius $\sqrt{2}$.

h) $\displaystyle\int_\gamma \frac{z^2\,dz}{e^{2\pi i z^3}-1}$, where γ is the circumference centered at 0 with radius r, where $n<r^3<n+1$ for some $n \in \mathbf{N}$.

i) $\displaystyle\int_\gamma \cosh z\,\mathrm{cotan}\, z\,dz$, where γ is the circumference centered at 0 with radius $(n+1/2)\pi$.

j) $\displaystyle\int_\gamma \frac{\sin(a/z)\,dz}{z^2+b}$ and $\displaystyle\int_\gamma \frac{\cos(a/z)\,dz}{z^2+b}$, where γ is the circumference centered at 0 with radius r, $a, b \in \mathbf{C}\setminus\{0\}$, $|b| \neq r^2$.

k) $\displaystyle\int_\gamma \frac{z^2+z^{-2}}{(\overline{z}-a)(b-\overline{z})}\,dz$, where γ is the circumference centered at 0 with radius r and $0<|a|<r<|b|$.

l) $\displaystyle\int_\gamma \Big(\sum_{n=-1}^{\infty} z^n\Big)\,dz$, where γ is the circumference centered at 0 with radius 1/2.

m) $\displaystyle\int_\gamma z^n\,e^{1/z}\,dz$, where $n \in \mathbf{N}$ and γ is any circumference surrounding 0.

n) $\displaystyle\int_\gamma (1+z+z^2)\,(e^{1/z}+e^{1/(z-1)}+e^{1/(z-2)})\,dz$, where γ is the circumference centered at 0 with radius 3.

o) $\displaystyle\int_\gamma P(z)\,(e^{1/z}+\cdots+e^{1/(z-k)})\,dz$, where γ is the circumference centered at 0 with radius $r>k$ and $P(z)$ is a polynomial of degree k.

p) $\displaystyle\int_\gamma \frac{dz}{|z-a|^2}$, where γ is the circumference centered at 0 with radius r, and $a \in \mathbf{C}$ with $|a| \neq r$.

Hints: k) $z\overline{z} = r^2$ if $z \in \gamma$, p) $|z-a|^2 = (z-a)(\overline{z}-\overline{a}) = (z-a)(r^2/z-\overline{a})$ if $z \in \gamma$.

Solutions: a) 0, b) $-\pi i\sqrt{2}/2$, c) 0, d) $12\pi i$, e) $-6\pi^2 i$, f) $2\pi i$, g) $-\pi i/2$, h) $2n+1$, i) $2\pi i\sum_{k=-n}^{n}\cosh k\pi$, j) $\frac{2\pi i}{\sqrt{b}}\sinh\frac{a}{\sqrt{b}}$ and 0, k) $\frac{2\pi i}{b-a}\frac{r^8+b^4}{b^4r^2}$, l) $2\pi i$, m) $2\pi i/(n+1)!$, n) $32\pi i$, o) $2\pi i\sum_{n=0}^{k}\frac{1}{n!(n+1)!}\sum_{m=0}^{k}P^{(n)}(m)$, p) $2\pi i a/(r^2-|a|^2)$, if $|a|<r$, $2\pi i a r^2/[|a|^2(|a|^2-r^2)]$, if $|a|>r$.

8.2. Compute the following integrals:

$$a)\ \int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx\,,\qquad b)\ \int_0^\infty \frac{x^2}{1+x^4}\,dx\,,\qquad c)\ \int_0^\infty \frac{dx}{x^6+1}\,,$$

$$d)\ \int_{-\infty}^\infty \frac{dx}{x^6+a^6}\,,\quad (a>0),\qquad e)\ \int_{-\infty}^\infty \frac{\cos x}{x^2-2x+2}\,dx\,,\qquad f)\ \int_{-\infty}^\infty \frac{\sin x}{x^2-2x+2}\,dx\,,$$

$$g)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2+1}\,dx\,,\qquad h)\ \int_{-\infty}^\infty \frac{\cos ax}{x^2+b^2}\,dx\,,\quad a,b>0,\qquad i)\ \int_{-\infty}^\infty \frac{\cos(\pi x/2)}{x^2-1}\,dx\,,$$

$$j)\ \text{p.v.}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx\,,\qquad k)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,dx\,,\qquad l)\ \int_{-\infty}^\infty \frac{\sin^3 x}{x^3}\,dx\,,$$

$$m)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2(1+x^2)}\,dx\,,\qquad n)\ \text{p.v.}\int_{-\infty}^\infty \frac{a\,\cos x + x\,\sin x}{x^2+a^2}\,dx\,,\quad a>0,$$

$$o)\ \int_0^\infty \frac{x^p}{x^n+1}\,,\quad n,p\in\mathbf{N},\ p<n-1,\qquad p)\ \int_0^\infty \frac{dx}{(a+bx^2)^n}\,,\quad n\in\mathbf{N},\ a,b>0\,.$$

Hints: *k*) Integrate the function $(1-e^{2ix})/x^2$.

l) Integrate the function $(-e^{3ix}+3e^{ix}-2)/x^3$.

m) Integrate the function $(1-e^{2ix})/(x^2(1+x^2))$.

Solutions: *a*) $\pi/4$, *b*) $\pi\sqrt{2}/4$, *c*) $\pi/3$, *d*) $2\pi/(3a^5)$, *e*) $\pi e^{-1}\cos 1$, *f*) $\pi e^{-1}\sin 1$, *g*) $\pi(1-e^{-2})/2$, *h*) $\pi e^{-ab}/b$, *i*) $-\pi$, *j*) π, *k*) π, *l*) $3\pi/4$, *m*) $\pi(1+e^{-2})/2$, *n*) $2\pi e^{-a}$, *o*) $\pi/[n\,\sin(\pi(p+1)/n)]$, *p*) $\frac{1\cdot3\cdot5\cdots(2n-3)}{2\cdot4\cdot6\cdots(2n-2)}\frac{\pi}{2}\frac{1}{a^n}\sqrt{\frac{a}{b}}$.

8.3. Compute the following integrals, if $a>0$ and $p\in(-1,1)$:

$$a)\ \int_0^\infty \frac{\log x}{x^2+a^2}\,dx\,,\qquad b)\ \int_0^\infty \frac{\sqrt{x}\,\log x}{x^2+a^2}\,dx\,,\qquad c)\ \int_0^\infty \frac{x^p}{x^2+a^2}\,dx\,,\qquad d)\ \int_0^\infty \frac{x^p\,\log x}{x^2+a^2}\,dx\,,$$

$$e)\ \int_0^\infty \frac{\log^2 x}{x^2+a^2}\,dx\,,\qquad f)\ \int_0^\infty \frac{\sqrt{x}\,\log^2 x}{x^2+a^2}\,dx\,,\qquad g)\ \int_0^\infty \frac{x^p\,\log^2 x}{x^2+a^2}\,dx\,,$$

$$h)\ \int_0^\infty \frac{\log x}{(x+a)^2}\,dx\,,\qquad i)\ \int_0^\infty \frac{\sqrt{x}\,\log x}{(x+a)^2}\,dx\,,\qquad j)\ \int_0^\infty \frac{x^p}{(x+a)^2}\,dx\,,\qquad k)\ \int_0^\infty \frac{x^p\,\log x}{(x+a)^2}\,dx\,,$$

$$l)\ \int_0^\infty \frac{\log^2 x}{(x+a)^2}\,dx\,,\qquad m)\ \int_0^\infty \frac{\sqrt{x}\,\log^2 x}{(x+a)^2}\,dx\,,\qquad n)\ \int_0^\infty \frac{x^p\,\log^2 x}{(x+a)^2}\,dx\,,$$

$$o)\ \int_0^\infty \frac{x\,\log^3 x}{(x+a)^2}\,dx\,,\qquad p)\ \int_0^\infty \frac{x^{-p}}{x+a}\,dx\,,\qquad q)\ \int_0^\infty \frac{x^{-p}\,\log x}{x+a}\,dx\,.$$

Hints: *a*), *b*), *c*), *d*), *e*), *f*), *g*), integrate along the boundary curve of the domain $\{z\in\mathbf{C}:\varepsilon<|z|<R,\,\mathrm{Im}\,z>0\}$. For the other items, you can integrate along the boundary curve of the domain $\{z\in\mathbf{C}:\varepsilon<|z|<R\}\setminus\{x+iy\in\mathbf{C}:\,x\ge0,\,|y|\le\delta\}$. *h*) integrate the function $(\log^2 z)/(z+a)^2$. *l*) integrate the function $(\log^3 z)/(z+a)^2$. *o*) is it an integrable function?

Solutions:

$$a)\ \frac{\pi\,\log a}{2\,a}\,,\qquad b)\ \frac{\pi}{2\sqrt{2\,a}}(\pi+2\,\log a)\,,\qquad c)\ \frac{\pi}{2\,a^{1-p}\cos(p\pi/2)}\,,$$

$$d)\ \frac{\pi}{4\,a^{1-p}\cos(p\pi/2)}(2\,\log a+\pi\,\tan(p\pi/2))\,,\qquad e)\ \frac{\pi}{8a}(\pi^2+4\,\log^2 a)\,,$$

$$f)\ \frac{\pi}{4\sqrt{2\,a}}(3\pi^2+4\pi\,\log a+4\,\log^2 a)\,,$$

$$g)\ \frac{\pi}{8\,a^{1-p}\cos(p\pi/2)}(\pi^2+4\,\log^2 a+4\pi\,\tan(p\pi/2)\,\log a+2\pi^2\tan^2(p\pi/2))\,,$$

$$h)\ \frac{\log a}{a}\,,\qquad i)\ \frac{\pi}{2\sqrt{a}}(2+\log a)\,,\qquad j)\ \frac{p\pi}{a^{1-p}\sin(p\pi)}\,,$$

$$k)\ \frac{p\pi}{a^{1-p}\sin(p\pi)}(\log a-\pi\text{cotan}\,(p\pi)+1/p)\,,\qquad l)\ \frac{\pi^2+3\,\log^2 a}{3a}\,,\qquad o)\ \infty\,,$$

$$p)\ \frac{\pi}{a^p\sin(p\pi)}\ \text{ if } p>0,\ \infty\ \text{ if } p\le 0,\qquad q)\ \frac{\pi}{a^p\sin(p\pi)}(\log a+\pi\,\text{cotan}\,(p\pi))\ \text{ if } p>0,\ \infty\ \text{ if } p\le 0\,.$$

8.4. Compute the following integrals:

$$a)\ \int_{-\infty}^{\infty}\frac{\sin(ax)}{\sinh x}\,dx\,,\quad a\in\mathbf{R}\qquad b)\ \int_{-\infty}^{\infty}\frac{\cos(ax)}{\cosh x}\,dx\,,\quad a\in\mathbf{R}\qquad c)\ \int_{-\infty}^{\infty}\frac{e^{ax}}{\cosh x}\,dx\,,\quad a\in(-1,1)\,,$$

$$d)\ \int_0^{\infty}\frac{\cosh(ax)}{\cosh x}\,dx\,,\quad a\in(-1,1)\,,\qquad e)\ \int_0^{\infty}\frac{\cosh(ax)}{\cosh(\pi x)}\,dx\,,\quad a\in(-\pi,\pi)\,,$$

$$f)\ \int_0^{\infty}\frac{x^2}{\cosh x}\,dx\,,\qquad g)\ \int_0^{\infty}\frac{x\cos(ax)}{\sinh x}\,dx\,.$$

Hints: *a)* $\sinh(z+2\pi i)=\sinh z$. *b)* $\cosh(z+\pi i)=-\cosh z$. *c)* Use item *b)*. *d)* Use item *c)*. *e)* Use item *d)*. *f)* Use item *d)* and integrate the function $x^3/\cosh x$.
Solutions:

$$a)\ \pi\,\tanh\frac{\pi\,a}{2}\,,\qquad b)\ \frac{\pi}{\cosh(\pi a/2)}\,,\qquad c)\ \frac{\pi}{\cos(\pi a/2)}\,,\qquad d)\ \frac{\pi}{2\cos(\pi a/2)}\,,$$

$$e)\ \frac{1}{2\cos(a/2)}\,,\qquad f)\ \frac{\pi^3}{8}\,,\qquad g)\ \frac{\pi^2 e^{\pi a}}{(e^{\pi a}+1)^2}\,.$$

8.5. Using that $\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt{\pi}$, prove that

$$\int_{-\infty}^{\infty}e^{-x^2}\cos 2bx\,dx=\sqrt{\pi}\,e^{-b^2}\,.$$

Hint: Integrate the function e^{-z^2} along an appropriate closed curve.

8.6. Compute the following integrals, if $a,\,b>0$:

$$a)\ \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta}\,,\ a^2>b^2+c^2\,,\quad b)\ \int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}\,,\ a>b>0\,,\quad c)\ \int_0^{2\pi}\frac{d\theta}{(a+b\cos^2\theta)^2}\,,$$

$$d)\ \int_0^{2\pi}e^{2\cos\theta}\,d\theta\,,\qquad e)\ \int_0^{2\pi}\frac{\cos(n\theta+\varphi)}{1-2a\cos\theta+a^2}\,d\theta\,,\quad a>1,\ n\in\mathbf{N}\,,\qquad f)\ \int_0^{2\pi}\cos^n\theta\,d\theta\,,\ n\in\mathbf{N}\,.$$

Solutions:

$$a)\ \frac{2\pi}{\sqrt{a^2-b^2-c^2}}\,,\qquad b)\ \frac{2\pi a}{(a^2-b^2)^{3/2}}\,,\qquad c)\ \frac{(2a+b)\pi}{[a(a+b)]^{3/2}}\,,\qquad d)\ 2\pi\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\,,$$

$$e)\ \frac{2n\cos\varphi}{a^n(a^2-1)}\,,\qquad f)\ \frac{\pi}{2^{n-1}}\binom{n}{n/2}\ \text{if } n \text{ is even,}\quad 0 \text{ if } n \text{ is odd.}$$

8.7. Prove that

$$\int_0^{\pi} e^{\cos\theta}\,\cos(\sin\theta)\,d\theta = \pi.$$

Hint: Use the complex integral

$$\int_{|z|=1} \frac{e^z}{z}\,dz\,.$$

8.8. Prove the following equalities, if $a \in \mathbf{R}$:

$$a)\ \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}\,,\qquad b)\ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,,\qquad c)\ \sum_{n=0}^{\infty} \frac{1}{a^2+n^2} = \frac{1}{2a^2}(1+\pi a\,\mathrm{cotanh}\,(\pi a))\,,$$

$$d)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{1}{2a^2}\Big(1+\frac{\pi a}{\sinh(\pi a)}\Big)\,,\qquad e)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}\,,$$

$$f)\ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!}\,,\qquad g)\ \sum_{n=1}^{\infty} \frac{1}{n^3}\,.$$

Remark: The last item is very complicated.

Hint: *f*) Use the Laurent series $\pi\,\mathrm{cotan}\,\pi z = \sum_{n=0}^{\infty}(-1)^n 2^{2n}\pi^{2n}B_{2n}z^{2n-1}/(2n)!$

8.9. Compute the Fresnel integrals

$$\int_0^{\infty} \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}\,,\qquad \int_0^{\infty} \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}\,,$$

which appear in diffraction theory.

Hint: Integrate the function $f(z) = e^{iz^2}$ along the boundary curve of the circular sector $\{z \in \mathbf{C} : |z| < R,\ \arg z \in (0, \pi/4)\}$. Use $\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$ and prove the inequality $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$.

8.10. For each $z \in \mathbf{C}$ with $\mathrm{Re}\, z > 0$, the Gamma function $\Gamma(z)$ is defined as $\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}\,dx$.

a) Prove that, for each $0 < a < b$, there exists an integrable function $f : (0,\infty) \to \mathbf{R}$ such that $|x^{z-1}e^{-x}| \le f(x)$ for every $x > 0$ and z with $a \le \mathrm{Re}\, z \le b$.

Hint: Consider separately the cases $0 < x \le 1$ and $x > 1$.

b) Prove that Γ is holomorphic on $\{\mathrm{Re}\, z > 0\}$.

Hint: Use Morera's theorem.

c) Prove that $\Gamma(z+1) = z\Gamma(z)$ on the set $\{z \in \mathbf{C} : \mathrm{Re}\, z > 0\}$.

Hint: Use integration by parts.

d) Prove that we have, for every $n \in \mathbf{N}$, $\Gamma(z+n+1) = (z+n)(z+n-1)\cdots(z+1)z\Gamma(z)$ on $\{\mathrm{Re}\, z > 0\}$. Deduce that $\Gamma(n+1) = n!$ for every $n \in \mathbf{N}$.

e) By using the previous item, for each z on $\{\mathrm{Re}\, z > -n-1\}$, we can define $\Gamma(z)$ as

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\cdots(z+1)z}\,.$$

Prove that Γ is well defined on $\mathbf{C}$, i.e., if $m > n$ and z satisfies $\mathrm{Re}\, z > -n-1$, then the corresponding definitions of $\Gamma(z)$ to n and m are the same.

f) The Gamma function is defined on the whole complex plane by the previous item. Prove that it is a meromorphic function on $\mathbf{C}$, and its unique poles are $\{0, -1, -2, \dots\}$. Prove also that each pole is simple and $\mathrm{Res}\,(\Gamma, -n) = (-1)^n/n!$.

Hint: Compute the limit $\lim_{z\to -n}(z+n)\Gamma(z)$.

8.11. a) The Beta function $B(z,w)$ is defined as

$$B(z,w) = \int_0^1 x^{z-1}\,(1-x)^{w-1}\,dx\,.$$

Prove that $x^{z-1}\,(1-x)^{w-1}$ is an integrable function on $(0,1)$ for any complex numbers z,w with $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, and that $B(z,w) = B(w,z)$.

b) Prove that this function is: b.1) holomorphic in the variable z on the domain $\{z : \operatorname{Re} z > 0\}$ for each w with $\operatorname{Re} w > 0$; b.2) holomorphic in the variable w on the domain $\{w : \operatorname{Re} w > 0\}$ for each z with $\operatorname{Re} z > 0$.

Hint: Use Morera's theorem and the symmetry of the Beta function.

c) Prove that

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}\,,$$

for any complex numbers z,w with $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$.

Hint: Prove that $\Gamma(z)\,\Gamma(w) = 4\int_0^\infty\int_0^\infty e^{-s^2-t^2}s^{2z-1}t^{2w-1}\,ds\,dt$ and use polar coordinates.

d) If $0 < a < 1$, prove that

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin(a\pi)}\,.$$

e) By making a change of variable in the integral in the previous item, prove the formula:

$$B(z,1-z) = \frac{\pi}{\sin(\pi z)}\,, \qquad 0 < z < 1\,.$$

Hint: Make the change of variable $t = 1/(x+1)$.

f) Prove that this formula holds for every $z \in \mathbf{C}\setminus\mathbf{Z}$, if we define $B(z,w)$ by the formula in item c) as a meromorphic function on the whole plane (on each variable).

Hint: If two holomorphic functions f,g on a domain Ω verify $f = g$ on a real interval, then $f = g$ on Ω (why?).

g) Prove that, for $b > a > -1$,

$$\int_0^{\pi/2} \cos^a t\,\cos bt\;dt \;=\; \frac{\pi\Gamma(a+1)}{2^{a+1}\,\Gamma(1+(a+b)/2)\,\Gamma(1+(a-b)/2)}\,.$$

Hint: Consider the integral $\int_\gamma (z+1/z)^a z^{b-1}\,dz$, where γ is the boundary of the domain $\{z \in \mathbf{C} : \operatorname{Re} z > 0,\ \varepsilon < |z| < 1,\ \varepsilon < |z-i|,\ \varepsilon < |z+i|\,\}$.

h) Does this formula hold if $a,\,b \in \mathbf{C}$ and $\operatorname{Re} a > -1$?

Hint: If two holomorphic functions f,g on a domain Ω verify $f = g$ on a real interval, then $f = g$ on Ω.