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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.4: Z-transform

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4 Z-transform

Problem 4.1 Find the Z-transform of the following functions (with sampling period T and $\alpha \in \mathbb{C}$):

$$\mathbf{1.}\ f(x) = 1\,, \qquad \mathbf{2.}\ f(x) = x\,,$$
$$\mathbf{3.}\ f(x) = x^2\,, \qquad \mathbf{4.}\ f(x) = x^3\,,$$
$$\mathbf{5.}\ f(x) = e^{\alpha x}\,, \qquad \mathbf{6.}\ f(x) = \cos(\alpha x)\,,$$

Solution: 1) $Z_T f(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-1/z} = \frac{z}{z-1}$.
2) $Z_T f(z) = \sum_{n=0}^{\infty} nTz^{-n} = zT\sum_{n=0}^{\infty} nz^{-n-1} = -zT\frac{d}{dz}\big(\sum_{n=0}^{\infty} z^{-n}\big) = zT\frac{1}{(z-1)^2} = \frac{zT}{(z-1)^2}$.
3) $Z_T f(z) = \sum_{n=0}^{\infty} n^2T^2z^{-n} = -Tz\frac{d}{dz}\big(\sum_{n=0}^{\infty} nTz^{-n}\big) = Tz\,\frac{2zT-T(z-1)}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$.
4) $Z_T f(z) = \sum_{n=0}^{\infty} n^3T^3z^{-n} = -Tz\frac{d}{dz}\big(\sum_{n=0}^{\infty} n^2T^2z^{-n}\big) = -Tz\,\frac{T^2(2z+1)(z-1)-3T^2(z+1)z}{(z-1)^4}$
$= -Tz\,\frac{2T^2z^2-T^2z-T^2-3T^2z^2-3T^2z}{(z-1)^4} = T^3\,\frac{z(z^2+4z+1)}{(z-1)^4}$.
5) $Z_T f(z) = \sum_{n=0}^{\infty} z^{-n}e^{\alpha nT} = \frac{1}{1-e^{\alpha T}/z} = \frac{z}{z-e^{\alpha T}}$.
6) $Z_T f(z) = \sum_{n=0}^{\infty} z^{-n}\cos(\alpha nT) = \frac{1}{2}\sum_{n=0}^{\infty} z^{-n}e^{i\alpha nT} + \frac{1}{2}\sum_{n=0}^{\infty} z^{-n}e^{-i\alpha nT}$
$= \frac{1}{2}\big(\frac{z}{z-e^{i\alpha T}} + \frac{z}{z-e^{-i\alpha T}}\big) = \frac{z(z-\cos(\alpha T))}{z^2-2z\cos(\alpha T)+1}$.

Problem 4.2 Find the Z-transform of the following sequences (with $a, b, \alpha, \beta \in \mathbb{C}$):

$$\mathbf{1.}\ f_n = 1\,, \qquad \mathbf{2.}\ f_n = n\,,$$
$$\mathbf{3.}\ f_n = n^2\,, \qquad \mathbf{4.}\ f_n = n^3\,,$$
$$\mathbf{5.}\ f_n = e^{\alpha n+\beta}\,, \qquad \mathbf{6.}\ f_n = a^n\,,$$
$$\mathbf{7.}\ f_n = \cos(\alpha n+\beta)\,, \qquad \mathbf{8.}\ f_n = \sin(\alpha n+\beta)\,,$$
$$\mathbf{9.}\ f_n = n\,a^n\,, \qquad \mathbf{10.}\ f_n = n^2a^n\,,$$
$$\mathbf{11.}\ f_n = n^3a^n\,, \qquad \mathbf{12.}\ f_n = a^{n-1}\,,\ \text{if}\ \ n \ge 1\,,\ f_0 = 0\,,$$
$$\mathbf{13.}\ f_n = \frac{a^n - b^n}{a-b}\,,\ a \ne b\,, \qquad \mathbf{14.}\ f_n = \frac{a^{n+1} - b^{n+1}}{a-b}\,,\ a \ne b\,.$$

Hint: Some Z-transforms can be deduced from the previous problem.
Solution: (1) $(Zf_n)(z) = \frac{z}{z-1}$, (2) $(Zf_n)(z) = \frac{z}{(z-1)^2}$, (3) $(Zf_n)(z) = \frac{z(z+1)}{(z-1)^3}$, (4) $(Zf_n)(z) = \frac{z(z^2+4z+1)}{(z-1)^4}$, (5) $(Zf_n)(z) = \frac{ze^{\beta}}{z-e^{\alpha}}$, (6) $(Zf_n)(z) = \frac{z}{z-a}$, (7) $(Zf_n)(z) = \frac{z^2\cos\beta - z\cos(\alpha-\beta)}{z^2-2z\cos\alpha+1}$, (8) $(Zf_n)(z) = \frac{z^2\sin\beta + z\sin(\alpha-\beta)}{z^2-2z\cos\alpha+1}$, (9) $(Zf_n)(z) = \frac{az}{(z-a)^2}$, (10) $(Zf_n)(z) = \frac{az(z+a)}{(z-a)^3}$, (11) $(Zf_n)(z) = \frac{az(z^2+4az+a^2)}{(z-a)^4}$, (12) $(Zf_n)(z) = \frac{1}{z-a}$, (13) $(Zf_n)(z) = \frac{z}{(z-a)(z-b)}$, (14) $(Zf_n)(z) = \frac{z^2}{(z-a)(z-b)}$.

Problem 4.3 Prove Theorem 4.3.
Hint: Use the properties of the Laurent series.

Problem 4.4 Prove Theorem 4.5.
Hint: Use the properties of the Laurent series.

Problem 4.5 Prove Theorem 4.6.
Hint: Use the properties of the Laurent series.

Problem 4.6 Find the Z-transform of the sequence f_n by using the items (1) and (2) in Theorem 2. Deduce the general term of the sequence f_n, by using some previous problem.

1. $f_{n+1} - 2f_n = 5$, with $f_0 = -4$,
2. $f_{n+2} - 2f_{n+1} + f_n = 0$, with $f_0 = 1$, $f_1 = 4$,
3. $f_{n+2} = f_{n+1} + f_n$, with $f_0 = 1$, $f_1 = 1$,
4. $f_{n+3} - 6f_{n+2} + 12f_{n+1} - 8f_n = 0$, with $f_0 = 0$, $f_1 = 2$, $f_2 = 16$
5. $f_{n+2} - f_{n+1} - 2f_n = 0$, with $f_0 = 0$, $f_1 = 1$,
6. $f_{n+1} + 3f_n = n$, with $f_0 = -2/3$,
7. $f_{n+2} - 3f_{n+1} + 2f_n = -1$, with $f_0 = -3/4$, $f_1 = 1/2$,
8. $4f_{n+2} + 4f_{n+1} + f_n = d_n$, with $d_0 = 1$, $d_n = 0$ $(n \geq 1)$, $f_0 = 0$, $f_1 = 1/4$.

Hint: Apply the Z-transform to both sides of each equation and use the formula relating $(Zf_{n+k})(z)$ with $(Zf_n)(z)$.
Solution: (1) $(Zf_n)(z) = \frac{z}{z-2} - 5\frac{z}{z-1}$, $f_n = 2^n - 5$. (2) $(Zf_n)(z) = \frac{z}{z-1} + 3\frac{z}{(z-1)^2}$, $f_n = 3n + 1$. (3) $(Zf_n)(z) = \frac{z^2}{z^2-z-1}$, $f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$. (4) $(Zf_n)(z) = \frac{2z(z+2)}{(z-2)^3}$, $f_n = n^2\, 2^n$. (5) $f_n = \frac{2^n}{3} + \frac{(-1)^{n+1}}{3}$. (6) $f_n = \frac{17}{16}(-3)^{n-1} - \frac{5}{16} + \frac{n}{4}$. (7) $f_n = n - 1 + 2^{n-2}$. (8) $f_0 = 0$, $f_n = (-1)^n(n-2)2^{-1-n}$ if $n \geq 1$.

A *digital filter* is an operator which maps a sequence f_n (called *input*) on other sequence g_n (called *output*) by using the formula

$$g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = \sum_{k=0}^{M} \alpha_k f_{n-k}\,, \qquad n \in \mathbb{N},$$

where $\alpha_k, \beta_k \in \mathbb{C}$ and $M, N \in \mathbb{N}$ are fixed constants, and $f_m = g_m = 0$ if $m < 0$.

Problem 4.7 Prove that if f_n and g_n are Z-transformable, then

$$(Zg_n)(z) = H(z)\,(Zf_n)(z)\,, \qquad \text{with} \qquad H(z) = \frac{\sum_{k=0}^{M} \alpha_k z^{-k}}{1 + \sum_{j=1}^{N} \beta_j z^{-j}}\,,$$

for z with large enough modulus. The function $H(z)$ is called digital filter *transfer function.*
Hint: Apply the Z-transform to both sides of the equation and relate $(Zg_{n-j})(z)$ and $(Zf_{n-k})(z)$ with $(Zg_n)(z)$ and $(Zf_n)(z)$, respectively.

Problem 4.8 The sequence h_n whose Z-transform is the transfer function H is called *shock response*. Prove that the output g_n is the convolution of h_n and the input f_n.
Hint: Use Theorem 4.5.

Problem 4.9 Consider the following digital filters with their inputs. In each case calculate the transfer function and the output:

1. $g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = \sum_{k=0}^{M} \alpha_k f_{n-k}$, with $f_n = 0$.

2. $g_n + \sum_{j=1}^{N} \beta_j g_{n-j} = f_n + \sum_{j=1}^{N} \beta_j f_{n-j}$, with any f_n.

3. $g_n + 3g_{n-1} + 2g_{n-2} = f_n$, with $f_0 = 1$, $f_n = 0$ if $n \geq 1$.

4. $g_n - g_{n-1} = f_n$, with $f_n = 1$.

5. $g_n + g_{n-1} = 3f_n + f_{n-1}$, with $f_n = 2n$.

Solution: (1) $H(z) = \frac{\sum_{k=0}^{M} \alpha_k z^{-k}}{1+\sum_{j=1}^{N} \beta_j z^{-j}}$, $(Zg_n)(z) = 0$, $g_n = 0$, (2) $H(z) = 1$, $(Zg_n)(z) = (Zf_n)(z)$, $g_n = f_n$, (3) $H(z) = \frac{z^2}{z^2+3z+2}$, $(Zg_n)(z) = \frac{z^2}{z^2+3z+2}$, $g_n = (-1)^{n+1} - (-2)^{n+1}$, (4) $H(z) = \frac{z}{z-1}$, $(Zg_n)(z) = \frac{z^2}{(z-1)^2}$, $g_n = n+1$, (5) $H(z) = \frac{3z+1}{z+1}$, $(Zg_n)(z) = \frac{6z^2+2z}{(z+1)(z-1)^2}$, $g_n = 4n + 1 - (-1)^n$.

Z-TRANSFORMS TABLE

$(a, b, \alpha, \beta \in \mathbb{C})$

$$Z[1](z) = \frac{z}{z-1}, \qquad Z[a^n](z) = \frac{z}{z-a},$$

$$Z[n](z) = \frac{z}{(z-1)^2}, \qquad Z[n\,a^n](z) = \frac{az}{(z-a)^2},$$

$$Z[n^2](z) = \frac{z(z+1)}{(z-1)^3}, \qquad Z[n^2\,a^n](z) = \frac{az(z+a)}{(z-a)^3},$$

$$Z[n^3](z) = \frac{z(z^2+4z+1)}{(z-1)^4}, \qquad Z[n^3\,a^n](z) = \frac{az(z^2+4az+a^2)}{(z-a)^4},$$

$$Z[\cos(\alpha n + \beta)](z) = \frac{z^2\cos\beta - z\cos(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1}, \qquad Z[\sin(\alpha n + \beta)](z) = \frac{z^2\sin\beta + z\sin(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1},$$

$$Z\Big[\frac{a^n - b^n}{a-b}\Big](z) = \frac{z}{(z-a)(z-b)} \quad (a \neq b), \qquad Z\Big[\frac{a^{n+1} - b^{n+1}}{a-b}\Big](z) = \frac{z^2}{(z-a)(z-b)} \quad (a \neq b),$$

$$Z[f_n](z) = \frac{1}{z-a}, \qquad \text{if } f_n = a^{n-1},\ n \geq 1,\ f_0 = 0.$$