COMPLEX VARIABLE AND TRANSFORMS | TEST 1 - SOLUTIONS

Problem 1 Find the radius of convergence of the series:

$$\sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \dots$$

We can write this series as $\sum_{k=0}^{\infty} a_k z^k$, with $a_k = 1$ if $k = 2^n$ for some $n \geq 0$ and $a_k = 0$ otherwise. Thus,

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{n\to\infty} |a_{2^n}|^{1/2^n} = \limsup_{n\to\infty} 1^{1/2^n} = \limsup_{n\to\infty} 1 = 1,$$

and so, $R = 1$.

Problem 2 Find the points in which the following series is convergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} z^{2n}.$$

Hint: Consider a change of variable.

With the change of variable $w = -z^2$, we have:

$$S = \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} z^{2n} = \sum_{n=2}^{\infty} \frac{(-z^2)^n}{\log n} = \sum_{n=2}^{\infty} \frac{1}{\log n} w^n.$$

The radius of convergence R_w of this new series is

$$\lim_{n\to\infty} \frac{\frac{1}{\log(n+1)}}{\frac{1}{\log n}} = \lim_{n\to\infty} \frac{\log n}{\log(n+1)} = 1 = \frac{1}{R_w},$$

and so, $R_w = 1$. Thus, S converges if $|w| < 1$ and diverges if $|w| > 1$. Since $1/\log n$ is a decreasing sequence with $\lim_{n\to\infty} 1/\log n = 0$, S converges if $|w| = 1$ with $w \neq 1$. If $w = 1$, then S diverges, since $\log n \leq n$ implies $\sum_{n=2}^{\infty} 1/\log n \geq \sum_{n=2}^{\infty} 1/n = \infty$.
Since $w = -z^2$, S converges if $|z| < 1$ and it diverges if $|z| > 1$; S converges if $|z| = 1$ with $z \neq \pm i$ and it diverges if $z = \pm i$.

Problem 3 Find the power series of the following function about 0:

$$\int_0^z \frac{\sin w}{w}\, dw.$$

Hint: Recall that

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, z^{2n+1}.$$

Since the radius of convergence of $\sin z$ is infinity, the following holds for every $z \in \mathbb{C}$:

$$\begin{aligned}\int_0^z \frac{\sin w}{w}\, dw &= \int_0^z \frac{1}{w} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, w^{2n+1} dw = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^z w^{2n} dw \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{w^{2n+1}}{2n+1}\right]_{w=0}^{w=z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)}\, z^{2n+1}.\end{aligned}$$

Problem 4 Compute the integral

$$\int_\gamma \frac{z}{1-\cos z}\, dz,$$

where γ is the circumference centered at 0 with radius 5, positively oriented.

Since z and $1-\cos z$ are holomorphic functions on the complex plane, its quotient is a holomorphic function on $\mathbb{C} \setminus \cup_{n\in\mathbb{Z}}\{2\pi n\}$, and it has a pole at each point $2\pi n$.
Since $z = 0$ is the only pole surrounded by γ, Residue Theorem gives

$$\int_\gamma \frac{z}{1-\cos z}\, dz = 2\pi i \ \mathrm{Res}\left(\frac{z}{1-\cos z}, 0\right).$$

The function $f(z) = 1-\cos z$ satisfies $f(0) = f'(0) = 0$, $f''(0) \neq 0$, and so, f has a zero of order 2 at $z = 0$. Hence, $z/(1-\cos z)$ has a pole of order 1 at $z = 0$, and

$$\begin{aligned}\int_\gamma \frac{z}{1-\cos z}\, dz &= 2\pi i \ \mathrm{Res}\left(\frac{z}{1-\cos z}, 0\right) = 2\pi i \lim_{z\to 0} \frac{z^2}{1-\cos z} \\ &= 2\pi i \lim_{z\to 0} \frac{2z}{\sin z} = 2\pi i \lim_{z\to 0} \frac{2}{\cos z} = 4\pi i.\end{aligned}$$

Problem 5 Compute the following integral, checking that the hypotheses (of the theorem that you use) hold:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx, \qquad a, b > 0.$$

There are simple poles at $z = \pm bi$ (since $z^2 + b^2$ is a polynomial with degree 2); $z = bi$ is the only pole with positive imaginary part (it is in the upper halfplane). Since $a > 0$, $1/(z^2 + b^2)$

is holomorphic on $\mathbb{C} \setminus \{0, \pm bi\}$ and $\lim_{z\to\infty} 1/(z^2+b^2) = 0$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx &= p.v. \int_{-\infty}^{\infty} \frac{\operatorname{Re}\left(e^{iax}\right)}{x^2+b^2}\, dx = \operatorname{Re}\left(p.v. \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\, dx \right) \\ &= \operatorname{Re}\left(2\pi i \operatorname{Res}\left(\frac{e^{iaz}}{z^2+b^2}, bi \right) \right) = \operatorname{Re}\left(2\pi i \lim_{z\to bi} \frac{(z-bi)\, e^{iaz}}{(z-bi)(z+bi)} \right) \\ &= \operatorname{Re}\left(2\pi i\, \frac{e^{-ab}}{2bi} \right) = \operatorname{Re}\left(\frac{\pi}{b}\, e^{-ab} \right) = \frac{\pi}{b}\, e^{-ab}.\end{aligned}$$

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