uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.2: Holomorphic functions

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1.2. HOLOMORPHIC FUNCTIONS

Definitions. *We denote by* $D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$ *the disk with center* z_0 *and radius* r.

Given $\Omega \subseteq \mathbf{C}$ *we say that:*

(1) Ω *is an* open *set if for every* $z_0 \in \Omega$*, there exists a disk* $D(z_0, r) \subseteq \Omega$.

(2) *An open set* Ω *is* connected *if for every* $z_1, z_2 \in \Omega$*, there exists a continuous curve* $\gamma : [a, b] \longrightarrow \Omega$ *from* z_1 *to* z_2*, i.e., such that* $\gamma(a) = z_1$, $\gamma(b) = z_2$.

(3) Ω *is a* region *or* domain *if it is an open connected set.*

(4) *A curve* $\gamma : [a, b] \to \mathbf{C}$ *is* closed *if* $\gamma(a) = \gamma(b)$.

(5) Ω *is* simply connected *if it is connected and every closed curve contained in* Ω *can be continuously transformed inside of* Ω *in a single point. In other words,* Ω *is connected and there is no closed curve in* Ω *surrounding some point which does not belong to* Ω *(i.e.,* Ω *does not have "holes").*

Given a function $f : \Omega \longrightarrow \mathbf{C}$, the concepts of limit and continuity coincide with those studied in Calculus for real functions if we consider $f : \Omega \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}^2$. Besides, the same results hold for additions, subtractions, products and quotients of functions.

Exercise. Show that if there exists $\lim_{z \to z_0} f(z) = w$, then

(a) $\lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} w$, (b) $\lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} w$,

(c) $\lim_{z \to z_0} \bar{f}(z) = \bar{w}$, (d) $\lim_{z \to z_0} |f(z)| = |w|$,

i.e., $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$ and $|z|$ are continuous functions. However, the argument is not a continuous function on $\mathbf{C}$, although it is continuous at $\mathbf{C}$ minus a ray starting at 0.

Exercise. Show that $\lim_{z \to z_0} f(z) = w$ if and only if $\lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} w$ and $\lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} w$.

Given a function $f : \Omega \longrightarrow \mathbf{C}$, in order to define the concept of derivative we can adopt the definition of differentiable function of two real variables $f : \Omega \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}^2$. However, it is more suitable to adopt the seemingly innocent strategy of defining the derivative in the "one-dimensional" way (as in the case of functions $f : \mathbf{R} \longrightarrow \mathbf{R}$) by using that $\mathbf{C}$ is a field (and so, we can divide complex numbers):

Definition. *Let* $\Omega \subseteq \mathbf{C}$ *be an open set,* $f : \Omega \longrightarrow \mathbf{C}$ *and* $z_0 \in \Omega$*. We say that* f *is* differentiable *at* z_0 *if there exists the limit (where* $z, h \in \mathbf{C}$)

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} .$$

Since this definition coincides with the differentiability of real functions of a real variable, we have the same rules for calculating derivatives (with the same proofs).

Theorem 1. *Let* f, g *be differentiable functions at* z_0*, and* $\alpha, \beta \in \mathbf{C}$*. Then:*

(1) $\alpha f + \beta g$ *is differentiable at* z_0*, and*

$$(\alpha f + \beta g)'(z_0) = \alpha\, f'(z_0) + \beta\, g'(z_0) .$$

(2) fg *is differentiable at* z_0*, and*

$$(fg)'(z_0) = f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0) .$$

(3) *If* $g(z_0) \neq 0$*,* f/g *is differentiable at* z_0*, and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\, g(z_0) - f(z_0)\, g'(z_0)}{g(z_0)^2} .$$

Theorem 2. *If* f *is differentiable at* z_0 *and* g *is differentiable at* $f(z_0)$*, then* $g \circ f$ *is differentiable at* z_0*, and the chain rule holds:*

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0) .$$

Theorem 3. *If* f *is differentiable at* z_0*, then* f *is continuous at* z_0.

EXAMPLES. The derivative of any constant function is 0, the derivative of $f(z) = z$ is 1, the derivative of $f(z) = z^n$ is $f'(z) = nz^{n-1}$, the derivative of $f(z) = 2z^3 - 7z^2 + 3z - 9$ is $f'(z) = 6z^2 - 14z + 3$.

EXAMPLES. Rational functions, i.e., the quotients of polynomials, are differentiable (except at the zeros of the denominator).

EXAMPLE. The derivative of $f(z) = z^2/(z+1)$ is $f'(z) = (z^2 + 2z)/(z+1)^2$.

EXAMPLE. The function $\bar{z}$ is not differentiable at any point.

Theorem 4. *If $f = u + iv$, the following statements are equivalent:*

(1) *f is differentiable at z_0.*

(2) *f is differentiable (in real sense) at z_0, and furthermore $f_y(z_0) = i f_x(z_0)$.*

(3) *u, v are differentiable (in real sense) at z_0 and the Cauchy-Riemann equations hold at z_0*

$$\begin{cases} u_x = v_y \,, \\ u_y = -v_x \,. \end{cases}$$

Furthermore, if f is differentiable at z_0, then $f'(z_0) = f_x(z_0)$.

PROOF (OF SOME IMPLICATIONS).

(1) $\implies$ (2) $\quad f_x(z_0) = f'(z_0)\dfrac{\partial z}{\partial x} = f'(z_0)\,; \quad f_y(z_0) = f'(z_0)\dfrac{\partial z}{\partial y} = i f'(z_0) = i f_x(z_0)\,.$

(2) $\iff$ (3) $\quad f_y = i f_x \iff u_y + i v_y = i(u_x + i v_x)$. ♯

Definition. *We say that f is holomorphic on an open set Ω if it is differentiable at every point in Ω. Also, we say that f is holomorphic at z_0 if it is holomorphic on some disk centered at z_0.*

EXAMPLES. If Ω is a domain and $f : \Omega \longrightarrow \mathbf{R}$ is differentiable at $z_0 \in \Omega$, we have $f'(z_0) = 0$; furthermore, if f is differentiable at every point in Ω, then $f' \equiv 0$ in Ω. Consequently, the constants are the only holomorphic functions on a domain which just take real values. Therefore, the functions $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\arg z$ are not holomorphic on any open set. The function $|z|^2$ is differentiable just at 0, and so, it is not holomorphic on any open set.

EXAMPLES. Using the Cauchy-Riemann equations it is easy to check that $f(z) = \bar{z}$ is not differentiable at any point in $\mathbf{C}$, but the complex exponential function is differentiable at every point in $\mathbf{C}$ if we define it as

$$f(z) = e^z := e^x e^{iy} := e^x(\cos y + i \sin y)\,, \qquad \text{and} \qquad f'(z) = e^z.$$

EXAMPLE. $f(z) = e^{z/(z^2+1)}$ is holomorphic on $\mathbf{C} \setminus \{-i, i\}$.

Theorem 5. *If f is holomorphic on an open set Ω, then f' is holomorphic on Ω.*

PROOF (IF f IS OF CLASS C^2 IN REAL SENSE). The function f' is of class C^1 on Ω and so, it is differentiable on Ω (in real sense). As f is differentiable on Ω, it satisfies the Cauchy-Riemann equations on Ω, and

$$f' = U + iV = u_x + i v_x \implies \begin{cases} U_x = (u_x)_x = (v_y)_x = (v_x)_y = V_y \,, \\ U_y = (u_x)_y = (u_y)_x = -(v_x)_x = -V_x \,. \end{cases}$$

Hence, f' satisfies the Cauchy-Riemann equations on Ω, and it is holomorphic on Ω. ♯

Definition. *If Ω is an open set in $\mathbf{R}^2$, a function $u : \Omega \longrightarrow \mathbf{R}$ of class C^2 is harmonic on Ω if*

$$\Delta u = u_{xx} + u_{yy} = 0 \qquad \text{on } \Omega\,.$$

Theorem 6. *If f is holomorphic on the open set Ω and it is of class C^2 on Ω (in real sense), then* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are harmonic functions on Ω.*

PROOF. Assume that $f = u + iv$. Let us prove that u is harmonic (the argument with v is similar).

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{yx} \end{cases} \implies u_{xx} + u_{yy} = 0\,. \quad \sharp$$

Definition. *u and v are harmonic conjugates on Ω (in this order) if $f = u + iv$ is holomorphic on Ω.*

If u and v are harmonic conjugates and $c_1, c_2 \in \mathbf{R}$, then $c_1 u$ and $c_1 v$ are harmonic conjugates, v and $-u$ (and $-v$ and u) are harmonics conjugates, and $u + c_1$ and $v + c_2$ are also harmonic conjugates.

We know that the real and imaginary parts of any holomorphic functions are harmonics functions. A natural question is the following one: are they the only harmonic functions? The answer is yes, at least for simply connected domains:

Theorem 7. *If Ω is a simply connected open set, then every harmonic function on Ω has a harmonic conjugate on Ω.*

Theorem 8. *If $f(z)$ is holomorphic at z_0 and $u(x, y)$ is its real part, then*

$$f(z) = 2\,u\Big(\frac{z + \overline{z_0}}{2}\,, \frac{z - \overline{z_0}}{2i}\Big) - \overline{f(z_0)}\,, \qquad \textit{where} \qquad \operatorname{Re} f(z_0) = u(z_0)\,.$$