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Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.5: Complex integration

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1.5. COMPLEX INTEGRATION

Integration along curves. All the curves $\gamma : [a,b] \longrightarrow \mathbf{C}$ considered from now on will be piecewise C^1 (γ is continuous on $[a,b]$, and $[a,b] = \cup_{k=1}^{n} [t_{k-1}, t_k]$ with γ' continuous on each $[t_{k-1}, t_k]$).

If $f : [a,b] \longrightarrow \mathbf{C}$ is a function with complex values, $f = u + iv$, and its real and imaginary parts u, v are integrable on $[a,b]$, the integral of f on $[a,b]$ is defined as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\,.$$

Recall that if $\gamma : [a,b] \longrightarrow \mathbf{C}$ is a curve, $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$, the integral along γ of a 1-form $P\,dx + Q\,dy$ (or the vector field (P,Q)) is defined as

$$\int_\gamma P\,dx + Q\,dy = \int_a^b \Big(P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)\Big)\,dt\,.$$

If f is defined on γ, i.e. $f : \gamma([a,b]) \longrightarrow \mathbf{C}$ and, for example, is continuous, then it makes sense to define

$$\int_\gamma f(z)\,dz = \int_\gamma f\,dz = \int_\gamma f\,dx + if\,dy\,.$$

Hence,

$$\int_\gamma f(z)\,dz = \int_\gamma \Big(f(\gamma(t))\,x'(t) + if(\gamma(t))\,y'(t)\Big)\,dt = \int_a^b f(\gamma(t))\,(x'(t) + iy'(t))\,dt\,,$$

and since $x'(t) + iy'(t) = \gamma'(t)$, we have

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt\,.$$

EXERCISES. $\int_\gamma \overline{z}\,dz$, where $\gamma(t) = t + 2ti$ for $t \in [0,1]$; $\int_{|z|=1} dz/z$.

Reparametrizations. If $t = t(s)$ is a <u>reparametrization</u> of γ, i.e., $t : [c,d] \longrightarrow [a,b]$ is C^1 and increasing, then $\gamma(t(s))$ is also a piecewise C^1 <u>parametrization</u> of the curve γ, and we have

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_c^d f(\gamma(t(s)))\,\gamma'(t(s))\,t'(s)\,ds = \int_c^d f((\gamma\circ t)(s))\,(\gamma\circ t)'(s)\,ds\,,$$

i.e., the definition of $\int_\gamma f$ does not depend on the parametrization of the curve γ, provided that we consider parameterizations with the same orientation.

Change of orientation. The opposite arc to $\gamma : [a,b] \longrightarrow \mathbf{C}$ is the arc $\tilde{\gamma} : [-b,-a] \longrightarrow \mathbf{C}$ defined as $\tilde{\gamma}(t) = \gamma(-t)$ and so, we have

$$\int_{\tilde{\gamma}} f(z)\,dz = \int_{-b}^{-a} f(\gamma(-t))\,(-\gamma'(-t))\,dt = -\int_a^b f(\gamma(t))\,\gamma'(t)\,dt = -\int_\gamma f(z)\,dz\,,$$

and, therefore, a change of orientation on the curve, produces a change of sign at the integral.

Integration with respect to arc-length. We define the integral of f with respect the arc-length on γ as

$$\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt = \int_\gamma f\,ds\,,$$

where s denotes the arc parameter. It is easy to check that this type of integration is independent both on changes of parametrization and also on orientation:

$$\int_{\tilde{\gamma}} f(z)\,|dz| = \int_{\gamma} f(z)\,|dz|\,.$$

We say that f is integrable along γ, and we write $f \in L^1(\gamma)$, if $\int_\gamma |f(z)|\,|dz| < \infty$.

Theorem 1. *Let γ, γ_i be curves and $\alpha, \beta \in \mathbf{C}$. If $\operatorname{length}\gamma = \int_\gamma |dz|$ denotes the length of γ, then:*

(1) $\displaystyle\int_\gamma (\alpha\, f + \beta\, g)\, dz = \alpha \int_\gamma f\, dz + \beta \int_\gamma g\, dz, \quad \forall\, f, g \in L^1(\gamma),$

(2) $\displaystyle\int_{\gamma_1 \cup \cdots \cup \gamma_n} f\, dz = \int_{\gamma_1} f\, dz + \cdots + \int_{\gamma_n} f\, dz,$ *if* $\operatorname{length}(\gamma_i \cap \gamma_j) = 0$ *for* $i \neq j$, $\forall\, f \in L^1(\gamma_1 \cup \cdots \cup \gamma_n)$,

(3) $\displaystyle\Big|\int_\gamma f(z)\, dz\Big| \leq \int_\gamma |f(z)|\,|dz|, \quad \forall\, f \in L^1(\gamma),$

(4) *if f is a measurable function on γ and $|f| \leq M$ on γ, then* $\big|\int_\gamma f\, dz\big| \leq M \operatorname{length}\gamma$.

Recall that a curve $\gamma : [a, b] \to \mathbf{C}$ is closed if $\gamma(a) = \gamma(b)$.

Theorem 2. *If $f : \Omega \longrightarrow \mathbf{C}$ is continuous on the open set Ω and f has a primitive on Ω, i.e., exists a holomorphic function F on Ω such that $f(z) = F'(z)$ for every $z \in \Omega$, then given any curve $\gamma : [a, b] \longrightarrow \Omega$, we have*

(1) $\displaystyle\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)),$

(2) $\displaystyle\int_\gamma f(z)\, dz = 0,$ *if γ is a closed curve.*

Proof. (1) is a consequence of the chain and Barrow rules:

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\,\gamma'(t)\, dt = \int_a^b F'(\gamma(t))\,\gamma'(t)\, dt = \int_a^b (F \circ \gamma)'(t)\, dt = F(\gamma(b)) - F(\gamma(a))\,.$$

Item (2) is a direct consequence of (1), since $\gamma(b) = \gamma(a)$. ♯

Example 1. $\displaystyle\int_\gamma e^z dz = 0$, for every closed curve γ.

Example 2. $\displaystyle\int_\gamma \frac{dz}{z^2} = 0$, for every closed curve γ that does not pass through the origin.

A simple closed curve is a curve whose starting point is also the end point and which has no other self-intersections.

Every simple closed curve in the complex plane admits a well-defined interior (and exterior); that follows from the Jordan curve theorem. A positively oriented curve in the complex plane is a simple closed curve such that when traveling on it one always has the curve interior to the left (and consequently, the curve exterior to the right), i.e. it is oriented counterclockwise (in the opposite direction to the movement of the hands of a clock).

Example 3. $\displaystyle\int_\gamma \frac{dz}{z - a} = 2\pi i$, if γ is a positively oriented curve with a in its interior such that γ intersects the line $\big\{z \in \mathbf{C} :\ \operatorname{Re} z = \operatorname{Re} a\big\}$ exactly twice:

Denote by α and β the two points in the intersection of γ and the line $\{\mathrm{Re}\, z = \mathrm{Re}\, a\}$ (with $\mathrm{Im}\,\alpha < \mathrm{Im}\,\beta$). Let $F_1(z) = \log(z-a) = \log|z-a| + i \arg(z-a)$, where we choose the argument $\arg(z-a) \in (-\pi, \pi]$. Since $F_1(z)$ is holomorphic on the halfplane $\{\mathrm{Re}\, z \geq \mathrm{Re}\, a\}$ except at the point a, and $F_1'(z) = 1/(z-a)$, we have for $\gamma_1 = \gamma \cap \{\mathrm{Re}\, z \geq \mathrm{Re}\, a\}$,

$$\int_{\gamma_1} \frac{dz}{z-a} = F_1(\beta) - F_1(\alpha) = \log|\beta - a| + i\pi/2 - \big(\log|\alpha - a| - i\pi/2\big) = \log \frac{|\beta - a|}{|\alpha - a|} + i\pi\,.$$

If we consider now the function $F_2(z) = \log(z-a) = \log|z-a| + i\arg(z-a)$, with $\arg(z-a) \in [0, 2\pi)$, then $F_2(z)$ is holomorphic on the halfplane $\{\mathrm{Re}\, z \leq \mathrm{Re}\, a\}$ except at the point a, and $F_2'(z) = 1/(z-a)$. If $\gamma_2 = \gamma \cap \{\mathrm{Re}\, z \leq \mathrm{Re}\, a\}$, then

$$\int_{\gamma_2} \frac{dz}{z-a} = F_2(\alpha) - F_2(\beta) = \log|\alpha - a| + i3\pi/2 - \big(\log|\beta - a| + i\pi/2\big) = -\log \frac{|\beta - a|}{|\alpha - a|} + i\pi\,.$$

Therefore,

$$\int_{\gamma} \frac{dz}{z-a} = \int_{\gamma_1} \frac{dz}{z-a} + \int_{\gamma_2} \frac{dz}{z-a} = \log \frac{|\beta - a|}{|\alpha - a|} + i\pi - \log \frac{|\beta - a|}{|\alpha - a|} + i\pi = 2\pi i\,. \quad \sharp$$

EXERCISE. What happens if a belongs to the exterior of the curve γ and there exists a halfline L starting at a with $L \cap \gamma = \emptyset$?

EXAMPLE 4. Since

$$\int_\gamma \frac{dz}{z} = \int_\gamma \frac{dx + i\,dy}{x + iy} = \int_\gamma \frac{x\,dx - iy\,dx + ix\,dy + y\,dy}{x^2 + y^2}\,,$$

we have

$$\mathrm{Im} \int_\gamma \frac{dz}{z} = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_\gamma d\Big(\arctan \frac{y}{x}\Big) = \int_\gamma d(\arg z)\,,$$

i.e., the imaginary part of the integral $\int_\gamma dz/z$ is the variation of the argument along γ.

Definition. *Let $\gamma : [\alpha, \beta] \longrightarrow \mathbf{C}$ be a closed curve (i.e., such that $\gamma(\alpha) = \gamma(\beta)$) and $a \notin \gamma([\alpha, \beta])$. The index or winding number of γ around the point a is the number $n(\gamma, a)$ defined by*

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}\,.$$

It can be proved that the winding number $n(\gamma, a)$ is an integer representing the total number of times that γ travels counterclockwisely around a. The winding number depends on the orientation of the curve, and is negative if γ travels around a clockwisely. The proof of this fact is very complicated, but it is easy to prove the following simpler result:

EXAMPLE 5. If γ is a closed curve of class C^1, then $n(\gamma, a) \in \mathbf{Z}$:

In order to prove that, consider $\gamma : [\alpha, \beta] \longrightarrow \mathbf{C}$ and the functions

$$h(x) = \int_\alpha^x \frac{\gamma'(t)}{\gamma(t) - a}\,dt\,, \qquad g(x) = e^{-h(x)}(\gamma(x) - a)\,,$$

defined when $x \in [\alpha, \beta]$. It is easy to check that g is differentiable on $[\alpha, \beta]$ and

$$g'(x) = -h'(x)\,e^{-h(x)}(\gamma(x) - a) + e^{-h(x)}\gamma'(x) = \frac{-\gamma'(x)}{\gamma(x) - a}\,e^{-h(x)}(\gamma(x) - a) + e^{-h(x)}\gamma'(x) = 0.$$

Therefore, g is constant on $[\alpha, \beta]$, and so,

$$e^{-h(\beta)}(\gamma(\beta) - a) = g(\beta) = g(\alpha) = e^{-h(\alpha)}(\gamma(\alpha) - a) = \gamma(\alpha) - a.$$

Since $\gamma(\beta) = \gamma(\alpha) \neq a$, we have $e^{-h(\beta)} = 1$. Therefore, $e^{2\pi i\, n(\gamma, a)} = e^{h(\beta)} = 1$ and so, $n(\gamma, a) \in \mathbf{Z}$. $\sharp$

Theorem 2 gives that if f has a primitive on Ω, then $\int_\gamma f = 0$ for every closed curve contained in Ω. The following result gives that both facts are equivalent.

Theorem 3. *Let $f : \Omega \longrightarrow \mathbf{C}$ be a continuous function on the open set Ω. The following facts are equivalent:*

(1) *There exists a holomorphic function $F : \Omega \longrightarrow \mathbf{C}$ such that $F' = f$.*

(2) $\displaystyle\int_\gamma f = 0$, *for every closed curve $\gamma \subset \Omega$.*

Proof. Theorem 2 gives that (1) implies (2).

Let us prove the converse implication. Fix $z_0 \in \Omega$. If Ω is connected, we can define the function

$$F(z) = \int_{z_0}^{z} f = \int_{\gamma_z} f\,,$$

where γ_z is any curve contained in Ω from z_0 to z, since $\int_\gamma f = 0$ for every closed curve $\gamma \subset \Omega$. It is easy to check (by using the argument in the proof of the Fundamental Theorem of Calculus) that $F' = f$ on Ω.

If Ω is not connected, then we can apply the previous argument to each connected component. ♯

Next, we show that $\int_\gamma f = 0$ holds for every closed curve γ and every holomorphic function f on Ω, if Ω is a simply connected set:

Theorem 4 (Cauchy's integral theorem). *If Ω is a simply connected open set and $f : \Omega \longrightarrow \mathbf{C}$ is holomorphic, then*

$$\int_\gamma f = 0\,, \qquad \text{for every closed curve } \gamma \subset \Omega\,.$$

Proof (if f' is continuous). If γ is a simple closed curve, let S be the interior of γ. Green's Theorem, with $P(x,y) = f(z)$, $Q(x,y) = if(z)$, gives

$$\int_\gamma f(z)\,dz = \int_{\partial S} f(z)\,dx + if(z)\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_S (if_x - f_y)\,dx\,dy = 0\,,$$

where the last equality follows from Cauchy-Riemann equations.

If γ is not simple, then it is the union of simple closed curves and we have proved that the integral along each of them is zero, and so, the integral along γ is zero. ♯

Proof (if γ is the boundary of a rectangle R). Given any rectangle Q contained in R, let us define

$$I(Q) = \int_{\partial Q} f(z)\,dz\,.$$

If we split the rectangle R into 4 congruent rectangles $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $R^{(4)}$, we have

$$I(R) = I(R^{(1)}) + I(R^{(2)}) + I(R^{(3)}) + I(R^{(4)})\,,$$

since the integrals over the common sides cancel each other. Consequently, there exists $R^{(k)}$, with $1 \le k \le 4$, such that

$$\left|I(R^{(k)})\right| \ge \frac{1}{4}\left|I(R)\right|.$$

Denote by R_1 this rectangle $R^{(k)}$.

If we repeat this process, then we obtain a sequence of rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$, satisfying

$$\left|I(R_n)\right| \ge \frac{1}{4^n}\left|I(R)\right|.$$

Denote by z_0 the point in R which is the limit of the sequence of rectangles $\{R_n\}$, i.e., $z_0 = \cap_n R_n$.

Fix any $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\Big|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\Big| < \varepsilon$$

for every z with $0 < |z - z_0| < \delta$, and so,

$$\big|f(z) - f(z_0) - f'(z_0)(z-z_0)\big| \le \varepsilon |z - z_0|$$

for every z with $|z - z_0| < \delta$.

Since $1 = (z)'$ and $z = (z^2/2)'$, we have

$$\int_{\partial R_n} dz = 0 = \int_{\partial R_n} z\, dz\,,$$

and so,

$$\begin{aligned}
\big|I(R_n)\big| &= \Big|\int_{\partial R_n} f(z)\, dz\Big| = \Big|\int_{\partial R_n} \big(f(z) - f(z_0) - f'(z_0)(z-z_0)\big)\, dz\Big| \\
&\le \int_{\partial R_n} \big|f(z) - f(z_0) - f'(z_0)(z-z_0)\big|\, |dz| \le \varepsilon \int_{\partial R_n} |z - z_0|\, |dz| \\
&< \varepsilon \int_{\partial R_n} \delta\, |dz| = \varepsilon\delta \operatorname{length}(\partial R_n) = \frac{1}{4^n}\, \varepsilon\delta \operatorname{length}(\partial R)\,, \\
\frac{1}{4^n}\, \big|I(R)\big| &\le \big|I(R_n)\big| \le \frac{1}{4^n}\, \varepsilon\delta \operatorname{length}(\partial R)\,, \\
\big|I(R)\big| &\le \varepsilon\delta \operatorname{length}(\partial R)\,,
\end{aligned}$$

and so, $\int_{\partial R} f = I(R) = 0$. ♯

Proof (γ is any closed curve). First of all, note that the previous proof has the following consequence: If $\sigma \subset \Omega$ is a finite union of horizontal and vertical segments such that it is the boundary of a polygon contained in Ω, then $\int_\sigma f = 0$.

Fix $z_0 \in \Omega$ and for each $z \in \Omega$ define

$$F(z) = \int_{z_0}^{z} f = \int_{\gamma_z} f\,,$$

where γ_z is any curve contained in Ω from z_0 to z obtained as a finite union of horizontal and vertical segments.

Let us check that F is well defined, i.e., it does not depend on the choice of γ_z: Assume that $\gamma_z, \eta_z \subset \Omega$ are curves from z_0 to z obtained as a finite union of horizontal and vertical segments. Therefore, $\gamma_z - \eta_z$ is a closed curve which is the boundary of a union of polygons $Q_1, \dots, Q_k$, and each ∂Q_j is a finite union of horizontal and vertical segments. Since Ω is simply connected, each Q_j is contained in Ω and so, $\int_{\partial Q_j} f = 0$. Hence,

$$0 = \sum_{j=1}^{k} \int_{\partial Q_j} f = \int_{\gamma_z - \eta_z} f = \int_{\gamma_z} f - \int_{\eta_z} f\,,$$

$$\int_{\gamma_z} f = \int_{\eta_z} f\,,$$

and F is well defined.

It is easy to check (by using the argument in the proof of the Fundamental Theorem of Calculus) that $F' = f$ on Ω. Then, Theorem 3 gives that $\int_\gamma f = 0$, for every closed curve $\gamma \subset \Omega$. ♯

Examples. If γ is any closed curve, we have

$$\int_\gamma e^{z^2}\, dz = 0\,.$$

If γ is any closed curve contained in the upper halfplane $\{z = x + iy \in \mathbf{C} :\ y > 0\}$, we have

$$\int_\gamma \frac{e^{z^2} + e^{\cos z}}{z+1}\, dz = 0\,.$$

It is easy to generalize the previous theorem to holomorphic functions with a finite set of singularities.

Theorem 5 (Cauchy's integral theorem with singularities). *If Ω is a simply connected open set, f is holomorphic on $\Omega \setminus \{a_1, \ldots, a_k\}$, and $\lim_{z\to a_j}(z - a_j)\, f(z) = 0$, for $j = 1, \ldots, k$, then*

$$\int_\gamma f = 0\,, \qquad \text{for every closed curve } \gamma \subset \Omega \setminus \{a_1, \ldots, a_k\}\,.$$