uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.7: Laurent series

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1.7. LAURENT SERIES

Definition. *A point $a \in \Omega$ is an* <u>*isolated singularity*</u> *of f, if f is holomorphic on $\Omega \setminus \{a\}$.*

If a is an isolated singularity of f, then there are only the three following possibilities:

(1) a is a <u>*removable singularity*</u> of f, i.e., $\lim_{z \to a}(z-a)\,f(z) = 0$. By Theorem 13 in the previous chapter, this happens if f is bounded on $D(a,r) \setminus \{a\}$ for some $r > 0$, or equivalently if

$$\exists \lim_{z \to a} f(z) \in \mathbf{C}\,.$$

(2) a is a <u>*pole*</u> of f, i.e.,

$$\lim_{z \to a} f(z) = \infty\,.$$

Proposition 1. *Let Ω be an open set, $a \in \Omega$ and f a holomorphic function on $\Omega \setminus \{a\}$. Then, a is a pole of f if and only if there exists an integer $k \geq 1$ (k is called the* <u>*order of the pole*</u> *of f at a) such that*

$$f(z) = \frac{g(z)}{(z-a)^k}\ ,$$

where g is a holomorphic function on Ω with $g(a) \neq 0$.

Proof. If there exists such a function g, then $\lim_{z \to a} f(z) = \infty$.

Conversely, if $\lim_{z \to a} f(z) = \infty$, then given $\alpha > 0$, there is $r > 0$ such that $|f(z)| > \alpha$, for every $z \in D(a,r) \setminus \{a\}$. Therefore, the function $F(z) = 1/f(z)$ is holomorphic and bounded on $D(a,r) \setminus \{a\}$, since $|F(z)| < 1/\alpha$ for every $z \in D(a,r) \setminus \{a\}$, and so, a is a removable singularity of F. If we define $F(a) := \lim_{z \to a} F(z) = \lim_{z \to a} 1/f(z) = 0$, then Corollary 8 in the previous chapter gives that $F(z) = (z-a)^k G(z)$, where $k \geq 1$, G is holomorphic on $D(a,r)$, and $G(a) \neq 0$. Hence,

$$f(z) = \frac{1}{F(z)} = \frac{1}{(z-a)^k G(z)} = \frac{g(z)}{(z-a)^k}\ ,$$

and, since $G(a) \neq 0$, $g(z) = 1/G(z)$ is holomorphic on a disk $D(a,\varepsilon)$ and $g(a) \neq 0$. Furthermore, $g(z) = (z-a)^k f(z)$ on $\Omega \setminus \{a\}$, and so, g is holomorphic on $\Omega \setminus \{a\}$. Thus, g is holomorphic on Ω. ♯

(3) Otherwise (i.e., (1) and (2) do not hold), a is an <u>*essential singularity*</u> of f.

Picard's great theorem gives that a is an essential singularity of f if and only if, for every $\varepsilon > 0$, $f(D(a,\varepsilon) \setminus \{a\})$ is the whole complex plane $\mathbf{C}$ unless, perhaps, a point. In particular, for every $w \in \mathbf{C}$, there exists a sequence $a_n \to a$ with $f(a_n) \to w$.

If a is a removable singularity of f, then f has a Taylor series about a

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

on some disk $D(a,r)$.

If f has on a a pole of order k, then Proposition 1 gives that f has a Laurent series

$$f(z) = \frac{c_{-k}}{(z-a)^k} + \cdots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n = \sum_{n=-k}^{\infty} c_n (z-a)^n, \qquad c_{-k} \neq 0\,,$$

in a "punctured" disk $D(a,r) \setminus \{a\}$.

Finally, if f has at a an essential singularity, then f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

with infinitely many c_{-n} ($n \in \mathbf{N}$) different from zero.

Example. 0 is an essential singularity of the function $f(z) = e^{1/z}$, since

$$e^{1/z} = \sum_{m=0}^{\infty} \frac{1}{m!}\Big(\frac{1}{z}\Big)^m = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}\,, \qquad z \neq 0\,.$$

Theorem 1. Assume that $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ are simple closed curves such that $\gamma_1, \gamma_2, \ldots, \gamma_n$ are contained in the interior of γ. If f is an holomorphic function on the closure of the domain Ω with $\partial\Omega = \cup_{k=1}^n \gamma_k \cup \gamma$, then

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\, dz\,,$$

and the following Cauchy's formula holds for every $a \in \Omega$:

$$\int_\gamma \frac{f(z)\, dz}{z-a} - \sum_{k=1}^{n} \int_{\gamma_k} \frac{f(z)\, dz}{z-a} = 2\pi i\, f(a)\,.$$

Theorem 2 (Laurent's theorem). *If f is holomorphic on $\Omega = \{z : r < |z-a| < R\}$ $(0 \le r < R \le \infty)$, then there is a sequence of complex numbers $\{c_n\}_{n=-\infty}^{\infty}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n\,, \qquad \forall\, z \in \Omega\,.$$

This series uniformly converges on compact sets contained in Ω. Furthermore, if Ω is the largest annulus centered on a such that f is holomorphic, then

$$r = \limsup_{n\to\infty} |c_{-n}|^{1/n}\,, \qquad \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}\,.$$

Proof. Fix $r < r_1 < \rho_1 < \rho_2 < r_2 < R$. We are going to prove that f is a Laurent series on the annulus $\{\rho_1 \le |z-a| \le \rho_2\}$. If $|w-a| = r_2$, then

$$\left|\frac{z-a}{w-a}\right| \le \frac{\rho_2}{r_2} < 1$$

and

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\,\frac{1}{1-\dfrac{z-a}{w-a}} = \frac{1}{w-a}\sum_{n=0}^{\infty}\Big(\frac{z-a}{w-a}\Big)^n = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}\,.$$

If $|w-a| = r_1$, then

$$\left|\frac{w-a}{z-a}\right| \le \frac{r_1}{\rho_1} < 1$$

and

$$\frac{1}{w-z} = \frac{-1}{z-a-(w-a)} = \frac{-1}{z-a}\,\frac{1}{1-\dfrac{w-a}{z-a}} = \frac{-1}{z-a}\sum_{n=0}^{\infty}\Big(\frac{w-a}{z-a}\Big)^n = -\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}\,.$$

If $\rho_1 \le |z-a| \le \rho_2$, then Theorem 1 gives

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|w-a|=r_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i}\int_{|w-a|=r_1} \frac{f(w)}{w-z}\, dw \\
&= \frac{1}{2\pi i}\int_{|w-a|=r_2} f(w)\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right) dw + \frac{1}{2\pi i}\int_{|w-a|=r_1} f(w)\left(\sum_{n=0}^{\infty}\frac{(w-a)^n}{(z-a)^{n+1}}\right) dw \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w-a|=r_2} \frac{f(w)}{(w-a)^{n+1}}\, dw\right)(z-a)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w-a|=r_1} f(w)(w-a)^n\, dw\right)\frac{1}{(z-a)^{n+1}}\,,
\end{aligned}$$

where we can change the order of integrals and series since the series converge uniformly.

Note that Theorem 1 gives that the values of the integrals

$$\frac{1}{2\pi i}\int_{|w-a|=r_2}\frac{f(w)}{(w-a)^{n+1}}\,dw\,,\qquad \frac{1}{2\pi i}\int_{|w-a|=r_1} f(w)(w-a)^n\,dw\,,$$

do not depend on r_1 and r_2, we can write

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n+\sum_{n=0}^{\infty}\frac{c_{-n-1}}{(z-a)^{n+1}}=\sum_{n=-\infty}^{\infty}c_n(z-a)^n$$

for every $\rho_1\le|z-a|\le\rho_2$. As we can choose any $r<r_1<\rho_1<\rho_2<r_2<R$, then f is equal to this Laurent series for every $z\in\Omega$. ♯

Exercise. Obtain the Laurent series of $f(z)=2/(z^2-4z+3)$, on the domains $\{|z|<1\}$, $\{1<|z|<3\}$ and $\{|z|>3\}$.

Definition. *We say that f is meromorphic on Ω if f is holomorphic on Ω unless on isolated singularities, and they are either removable or poles.*

Corollary 1 (Bernouilli-l'Hôpital's rule 3). *If f_1 and f_2 are meromorphic functions and have a pole at a, then*

$$\lim_{z\to a}\frac{f_1(z)}{f_2(z)}=\lim_{z\to a}\frac{f_1'(z)}{f_2'(z)}\,.$$

Proof. Let k_j be the order of the pole of f_j at a, then $f_j(z)=g_j(z)/(z-a)^{k_j}$ with g_j holomorphic at a and $g_j(a)\neq0$ for $k=1,2$.

If $k_1=k_2$, then

$$\lim_{z\to a}\frac{f_1(z)}{f_2(z)}=\lim_{z\to a}\frac{g_1(z)(z-a)^{-k_1}}{g_2(z)(z-a)^{-k_1}}=\lim_{z\to a}\frac{g_1(z)}{g_2(z)}=\frac{g_1(a)}{g_2(a)}$$

and

$$\lim_{z\to a}\frac{f_1'(z)}{f_2'(z)}=\lim_{z\to a}\frac{g_1'(z)(z-a)^{-k_1}-k_1g_1(z)(z-a)^{-k_1-1}}{g_2'(z)(z-a)^{-k_1}-k_1g_2(z)(z-a)^{-k_1-1}}=\lim_{z\to a}\frac{g_1'(z)(z-a)-k_1g_1(z)}{g_2'(z)(z-a)-k_1g_2(z)}=\frac{g_1(a)}{g_2(a)}\,.$$

If $k_1<k_2$, then

$$\lim_{z\to a}\frac{f_1(z)}{f_2(z)}=\lim_{z\to a}\frac{g_1(z)(z-a)^{-k_1}}{g_2(z)(z-a)^{-k_2}}=\lim_{z\to a}(z-a)^{k_2-k_1}\frac{g_1(z)}{g_2(z)}=0$$

and

$$\lim_{z\to a}\frac{f_1'(z)}{f_2'(z)}=\lim_{z\to a}\frac{g_1'(z)(z-a)^{-k_1}-k_1g_1(z)(z-a)^{-k_1-1}}{g_2'(z)(z-a)^{-k_2}-k_2g_2(z)(z-a)^{-k_2-1}}=\lim_{z\to a}(z-a)^{k_2-k_1}\frac{g_1'(z)(z-a)-k_1g_1(z)}{g_2'(z)(z-a)-k_2g_2(z)}=0.$$

If $k_1>k_2$, then a similar argument gives the result. ♯

Corollary 2 (Bernouilli-l'Hôpital's rule 4). *Let f_1, f_2 be two holomorphic functions on $\{|z|>r\}$ for some $r>0$, such that $\lim_{z\to\infty}f_1(z)=\lim_{z\to\infty}f_2(z)=\infty$. Then*

$$\lim_{z\to\infty}\frac{f_1(z)}{f_2(z)}=\lim_{z\to\infty}\frac{f_1'(z)}{f_2'(z)}\,.$$

Proof. The functions $f_1(1/z)$ and $f_2(1/z)$ are meromorphic on a neighborhood of 0 (and they have a pole at 0 since $\lim_{z\to0}f_1(1/z)=\lim_{z\to0}f_2(1/z)=\infty$). Hence, they satisfies the hypotheses on Bernouilli-l'Hôpital's rule 3 at 0 and

$$\lim_{z\to\infty}\frac{f_1(z)}{f_2(z)}=\lim_{w\to0}\frac{f_1(1/w)}{f_2(1/w)}=\lim_{w\to0}\frac{-w^{-2}f_1'(1/w)}{-w^{-2}f_2'(1/w)}=\lim_{z\to\infty}\frac{f_1'(z)}{f_2'(z)}\,.\qquad ♯$$

Theorem 3. *f is meromorphic on the open set Ω if and only if $f=g/h$ with g,h holomorphic functions on Ω and h is not identically zero.*