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Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.8: Residue theorem

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1.8. THE RESIDUE THEOREM AND ITS APPLICATIONS

Definition. *If a is an isolated singularity of f, the residue of f at $z = a$ is the coefficient c_{-1} of its Laurent series about $z = a$. We write* $\operatorname{Res}(f, a) = c_{-1}$.

If a is a pole of order 1 of f (i.e., a simple pole), then

$$\operatorname{Res}(f,a) = \lim_{z\to a}(z-a)f(z)\,.$$

If a is a pole of order k of f, then

$$\operatorname{Res}(f,a) = \frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}\Big((z-a)^k f(z)\Big).$$

If f has at a an essential singularity, there is NO formula for the residue of f at a; there is only one way to find this residue: to find the Laurent series of f about a and to find the coefficient c_{-1} of the power $(z-a)^{-1}$ in this Laurent series (this method also works if a is a pole of f).

If f has an isolated singularity at a and there exists $\lim_{z\to a}(z-a)f(z) = \ell$, with $\ell \neq 0, \infty$, then a is a pole of order 1 of f and its residue is ℓ.

Corollary. *If $z_1, \dots, z_n$ are different complex numbers and $P(z)$ is a polynomial with degree less than n, then the following partial fraction decomposition holds*

$$Q(z) = \frac{P(z)}{(z-z_1)\cdots(z-z_n)} = \frac{\operatorname{Res}(Q,z_1)}{z-z_1} + \cdots + \frac{\operatorname{Res}(Q,z_n)}{z-z_n}\,, \qquad \textit{with} \quad \operatorname{Res}(Q,z_j) = \lim_{z\to z_j}(z-z_j)Q(z)\,.$$

Exercise. Find the partial fraction decomposition of $\dfrac{1}{z^3-z}$.

Residue theorem. *Let Ω be a simply connected open set, f a holomorphic function on Ω unless on a set $\{a_j\}$ of isolated singularities, and $\gamma \subset \Omega$ a closed curve that does not go through any a_j. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}(f,a_j)\, n(\gamma,a_j)\,.$$

Proposition 1.

$$\int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta = \int_0^{2\pi} R\Big(\frac{e^{i\theta}+e^{-i\theta}}{2}\,,\frac{e^{i\theta}-e^{-i\theta}}{2i}\Big)\,\frac{ie^{i\theta}d\theta}{ie^{i\theta}} = \int_{|z|=1} R\Big(\frac{1}{2}\Big(z+\frac{1}{z}\Big)\,,\frac{1}{2i}\Big(z-\frac{1}{z}\Big)\Big)\,\frac{dz}{iz}\,.$$

Remark. In order to apply Proposition 1, it is a VERY BAD IDEA to use the formulas $\cos\theta = \operatorname{Re} e^{i\theta}$ and $\sin\theta = \operatorname{Im} e^{i\theta}$, since the real and imaginary parts are not holomorphic functions.

Exercises. Compute $\displaystyle\int_0^{2\pi} \sin^n\theta\,d\theta$ and $\displaystyle\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta$.

In the following result, p.v. denotes the Cauchy principal value of an integral. Recall that if the poles of f on the real line are $x_1 < \cdots < x_n$, then the Cauchy principal value of $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0^+}}\Big(\int_{-R}^{x_1-\varepsilon} f(x)\,dx + \int_{x_1+\varepsilon}^{x_2-\varepsilon} f(x)\,dx + \cdots + \int_{x_{n-1}+\varepsilon}^{x_n-\varepsilon} f(x)\,dx + \int_{x_n+\varepsilon}^{R} f(x)\,dx\Big)\,,$$

and it is equal to the integral $\int_{-\infty}^{\infty} f(x)\,dx$ if f is integrable.

Proposition 2. *Let f be an holomorphic function on a domain containing the closure of the upper halfplane $\mathbf{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, unless at a finite number of singularities, which can belong to the real axis, but in this case they must be simple poles.*

(1) *If $\lim_{z\to\infty} z\, f(z) = 0$, then*

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ \operatorname{Im} a>0}} \operatorname{Res}(f,a) + \pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ a\in \mathbf{R}}} \operatorname{Res}(f,a)\,.$$

(2) *If $\lim_{z\to\infty} f(z) = 0$, and $c > 0$, then*

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, e^{icx}\, dx = 2\pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ \operatorname{Im} a>0}} \operatorname{Res}\big(f(z)\, e^{icz}, a\big) + \pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ a\in \mathbf{R}}} \operatorname{Res}\big(f(z)\, e^{icz}, a\big)\,.$$

(3) *If $\lim_{z\to\infty} z\, f(z) = 0$, $k_0, k_1, \ldots, k_n \in \mathbf{C}$, $c_1, \ldots, c_n > 0$, and $g(z) = k_0 + k_1\, e^{ic_1 z} + \cdots + k_n\, e^{ic_n z}$, then*

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, g(x)\, dx = 2\pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ \operatorname{Im} a>0}} \operatorname{Res}(fg,a) + \pi i \sum_{\substack{a\in \mathrm{Sing}(f)\\ a\in \mathbf{R}}} \operatorname{Res}(fg,a)\,.$$

Remarks to Proposition 2.

(a) If f has poles of order greater than 1 in the real axis, but these poles are removable singularities or simple poles of fg, then the conclusion of (3) also holds.

(b) If $\lim_{z\to\infty} z\, f(z) = 0$ and f does not have poles in the real axis, then the functions in the integrals in (1), (2) and (3) are integrable, and so, their integrals are equal to their principal values.

(c) If $k_0 = 0$, the conclusion of (3) also holds if we replace the hypothesis $\lim_{z\to\infty} zf(z) = 0$ by the weaker one $\lim_{z\to\infty} f(z) = 0$.

(d) Note that if f has a non-simple pole in the real axis, then $f \notin L^1(\mathbf{R})$ and $\text{p.v.}\int_{-\infty}^{\infty} f(x)\, dx$ does not exist.

(e) Although in general

$$\operatorname{Re}(zw) \neq (\operatorname{Re} z)(\operatorname{Re} w), \qquad \operatorname{Re}(z^2) \neq (\operatorname{Re} z)^2, \qquad \frac{1}{\operatorname{Re} z} \neq \operatorname{Re}\frac{1}{z}\,,$$

we have $\operatorname{Re}(az) = a\operatorname{Re} z$ for every $a \in \mathbf{R}$ (and the same holds for the imaginary part). Hence, if f takes real values on the real line and $c > 0$,

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\, \cos cx\, dx = \text{p.v.}\int_{-\infty}^{\infty} f(x) \operatorname{Re}(e^{icx})\, dx = \operatorname{Re}\Big(\text{p.v.}\int_{-\infty}^{\infty} f(x)\, e^{icx}\, dx\Big),$$
$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\, \sin cx\, dx = \text{p.v.}\int_{-\infty}^{\infty} f(x) \operatorname{Im}(e^{icx})\, dx = \operatorname{Im}\Big(\text{p.v.}\int_{-\infty}^{\infty} f(x)\, e^{icx}\, dx\Big).$$

(f) If f takes real values on the real line and $c < 0$, then $-c > 0$ and

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\, e^{icx}\, dx = \overline{\text{p.v.}\int_{-\infty}^{\infty} f(x)\, e^{-icx}\, dx}.$$

(g) In order to apply Proposition 2, it is a VERY BAD IDEA to use the formulas $\cos cx = \frac{1}{2}(e^{icx} + e^{-icx})$ and $\sin cx = \frac{1}{2i}(e^{icx} - e^{-icx})$, since $-c < 0$ if $c > 0$ and vice versa.

(h) If f takes real values on the real line and $c > 0$, then

$$\begin{aligned}
\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\cos^2 cx\,dx &= \text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\frac{1}{2}\,(1+\cos 2cx)\,dx = \frac{1}{2}\,\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\mathrm{Re}\,(1+e^{i2cx})\,dx \\
&= \frac{1}{2}\,\mathrm{Re}\left(\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,(1+e^{i2cx})\,dx\right), \\
\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\sin^2 cx\,dx &= \text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\frac{1}{2}\,(1-\cos 2cx)\,dx = \frac{1}{2}\,\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,\mathrm{Re}\,(1-e^{i2cx})\,dx \\
&= \frac{1}{2}\,\mathrm{Re}\left(\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,(1-e^{i2cx})\,dx\right).
\end{aligned}$$

(i) It is a BAD IDEA to use the formula

$$\begin{aligned}
\text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,(k_0 + k_1\,e^{ic_1x} + \cdots + k_n\,e^{ic_nx})\,dx &= \text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,k_0\,dx + \text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,k_1\,e^{ic_1x}\,dx + \\
&\cdots + \text{p.v.}\!\int_{-\infty}^{\infty} f(x)\,k_n\,e^{ic_nx}\,dx,
\end{aligned}$$

since perhaps some of the principal values in the right side of the formula do not exist.

Example. Compute $\text{p.v.}\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$.

Proposition 3. *Let $\alpha \in \mathbf{R}\backslash\mathbf{Z}$, and f a holomorphic function on $\mathbf{C}$ except at a finite number of singularities, which are not in the positive real axis $\mathbf{R}^+ = [0,\infty)$, and such that*

$$\lim_{z\to 0} z^{\alpha+1} f(z) = 0\,, \qquad \lim_{z\to\infty} z^{\alpha+1} f(z) = 0\,.$$

Then $x^\alpha f(x)$ is integrable on $[0,\infty)$ and

$$\int_0^\infty x^\alpha f(x)\,dx = \frac{2\pi i}{1-e^{2\pi\alpha i}} \sum_{a\in \mathrm{Sing}(f)} \mathrm{Res}\left(z^\alpha f(z),\, a\right),$$

where we consider the argument in the interval $[0,2\pi)$ in order to define z^α.

Example. Compute for appropriate values of $\alpha \in \mathbf{R}$:

$$\int_0^\infty \frac{x^\alpha}{x^2+1}\,dx\,.$$

The following result allows to compute the integral of very general functions on $(0,\infty)$.

Proposition 4. *Let f be a holomorphic function on $\mathbf{C}$ except at a finite number of singularities, which are not in the positive real axis $\mathbf{R}^+ = [0,\infty)$, and such that $\lim_{z\to\infty} z\,f(z) = 0$. Then*

$$\int_0^\infty f(x)\,dx = -\sum_{a\in \mathrm{Sing}(f)} \mathrm{Res}\left(f(z)\log z, a\right),$$

where we consider the argument in the interval $[0,2\pi)$ in order to define $\log z$.

Example. Compute $\displaystyle\int_0^\infty \frac{dx}{x^3+1}$.

Finally, the following proposition allows to compute the value of many series as another consequence of the residue theorem.

Proposition 5. *Let f be a holomorphic function on $\mathbf{C}$ except at a finite number of singularities and such that* $\lim_{z\to\infty} z\,f(z) = 0$. *Then*

$$(1) \qquad \sum_{\substack{n=-\infty \\ n\notin \mathrm{Sing}(f)}}^{\infty} f(n) = -\pi \sum_{a\in \mathrm{Sing}(f)} \mathrm{Res}\big(f(z)\,\mathrm{cotan}\,\pi z\,,\,a\big)\,.$$

$$(2) \qquad \sum_{\substack{n=-\infty \\ n\notin \mathrm{Sing}(f)}}^{\infty} (-1)^n f(n) = -\pi \sum_{a\in \mathrm{Sing}(f)} \mathrm{Res}\big(f(z)\,\mathrm{cosec}\,\pi z\,,\,a\big)\,.$$

Example. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$