uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 2: Transforms

Section 2.4: Z-transform

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

4 Z-transform

If $f : [0, \infty) \longrightarrow \mathbb{C}$, we define the discrete Laplace transform with sampling period $T > 0$ as the function (of the complex variable w) given by

$$(L_T f)(w) = \sum_{n=0}^{\infty} f(nT)\, e^{-nTw} \,.$$

If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers, its discrete Laplace transform can also be defined by means of the above formula, by substituting $f(nT)$ and T, by f_n and 1, respectively, i.e.,

$$(L\{f_n\})(w) = (Lf_n)(w) = \sum_{n=0}^{\infty} f_n\, e^{-nw} \,.$$

In both cases the Z-transform is defined as

$$(Z_T f)(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \,, \qquad (Zf_n)(z) = \sum_{n=0}^{\infty} f_n\, z^{-n} \,.$$

Therefore, $(L_T f)(w) = (Z_T f)(e^{Tw})$ and $(Lf_n)(w) = (Zf_n)(e^w)$.

As a direct consequence of Cauchy-Hadamard formula, we have:

Theorem 4.1 *If $f : [0, \infty) \longrightarrow \mathbb{C}$, then $Z_T f$ and Zf_n converge if $|z| > \rho_T$ and if $|z| > \rho$, respectively, with*

$$\rho_T = \limsup_{n\to\infty} |f(nT)|^{1/n}, \qquad \rho = \limsup_{n\to\infty} |f_n|^{1/n}.$$

Definition 4.2 *We say that f is Z_T-transformable if $\rho_T < \infty$. Analogously, we say that f_n is Z-transformable if $\rho < \infty$.*

The next result contains the main properties of the Z-transform.

Theorem 4.3 *If f_n, g_n are Z-transformable sequences, $\alpha, \beta \in \mathbb{C}$ and $k \in \mathbb{N}$, we have:*

(1) $Z(\alpha f_n + \beta g_n) = \alpha\, Zf_n + \beta\, Zg_n$.

(2) $(Zf_{n+k})(z) = z^k\Big((Zf_n)(z) - \sum_{j=0}^{k-1} f_j z^{-j}\Big)$.

(3) $(Zf_{n-k})(z) = z^{-k}(Zf_n)(z)$, *if we define* $f_m = 0$ *for* $m < 0$.

(4) $(Z(\alpha^n f_n))(z) = (Zf_n)(z/\alpha)$, *if* $\alpha \neq 0$.

(5) $(Z(nf_n))(z) = -z\,\dfrac{d}{dz}(Zf_n)(z)$.

(6) *If $(Zf_n)(z)$ converges for $|z| > \rho$, g_n is the sequence $g_n = f_n/n$ if $n \geq 1$ (g_0 can be any number), and $|z| > \rho$, then*

$$(Zg_n)(z) = g_0 + \int_z^{\infty} \frac{(Zf_n)(\xi) - f_0}{\xi}\, d\xi \,.$$

(7) *If we define the first difference of the sequence f_n as $\Delta f_n = f_{n+1} - f_n$, then*

$$(Z(\Delta f_n))(z) = (z-1)(Zf_n)(z) - f_0 z \,.$$

(8) *If we define the second difference of the sequence* f_n *as*

$$\Delta^2 f_n = \Delta(\Delta f_n) = \Delta f_{n+1} - \Delta f_n = f_{n+2} - 2f_{n+1} + f_n \,,$$

then

$$\begin{aligned}(Z(\Delta^2 f_n))(z) &= (z-1)^2 (Zf_n)(z) - f_0 z(z-1) - (f_1 - f_0)z \\ &= (z-1)^2 (Zf_n)(z) - f_0 z(z-1) - \Delta f_0 z\,.\end{aligned}$$

(9) *If we define the* k*-th difference of the sequence* f_n *as*

$$\Delta^k f_n = \Delta(\Delta^{k-1} f_n) = \Delta^{k-1} f_{n+1} - \Delta^{k-1} f_n \,,$$

then

$$\big(Z(\Delta^k f_n)\big)(z) = (z-1)^k (Zf_n)(z) - z\sum_{j=0}^{k-1} (z-1)^{k-j-1} \Delta^j f_0 \,,$$

where $\Delta^0 f_0 = f_0$ *and* $\Delta^1 f_0 = \Delta f_0$.

(10) *If* $s_n = \sum_{k=0}^{n} f_k$, Zf_n *converges on* $|z| > \rho$, *and* $|z| > \max\{1, \rho\}$, *then* $(Zs_n)(z) = z\,(Zf_n)(z)/(z-1)$.

(11) *If* $t_n = \sum_{k=0}^{n-1} f_k$ *for* $n \geq 1$, Zf_n *converges on* $|z| > \rho$, *and* $|z| > \max\{1, \rho\}$, *then* $(Zt_n)(z) = t_0 + (Zf_n)(z)/(z-1)$.

(12) *If* $f_m = 0$ *when* m *is not an integer multiple of* k, *define* $g_n = f_{kn}$. *If* Zf_n *converges on* $|z| > \rho$, *then* Zg_n *converges on* $|z| > \rho^k$ *and* $(Zg_n)(z) = (Zf_n)(z^{1/k})$.

(13) *Define* $g_m = 0$ *when* m *is not an integer multiple of* k, *and* $g_{kn} = f_n$. *If* Zf_n *converges on* $|z| > \rho$, *then* Zg_n *converges on* $|z| > \rho^{1/k}$ *and* $(Zg_n)(z) = (Zf_n)(z^k)$.

Definition 4.4 *Assume that the Z-transforms of the sequences* f_n *and* g_n *converge, respectively, on* $|z| > \rho_f$ *and* $|z| > \rho_g$. *The* <u>*convolution*</u> *of the sequences* f_n *and* g_n *is defined as the sequence* h_n *given by*

$$h_n = f_n * g_n = \sum_{k=0}^{n} f_k\, g_{n-k} \,.$$

Theorem 4.5 *Consider sequences* f_n, g_n *and* h_n. *Then:*

a) $f_n * g_n = g_n * f_n$.
b) $f_n * (g_n + h_n) = f_n * g_n + f_n * h_n$.
c) $f_n * (g_n * h_n) = (f_n * g_n) * h_n$.

Theorem 4.6 *Assume that the Z-transforms of the sequences* f_n *and* g_n *converge on* $|z| > \rho_f$ *and* $|z| > \rho_g$, *respectively. If* $|z| > \max\{\rho_f, \rho_g\}$, *then*

$$(Z(f_n * g_n))(z) = (Zf_n)(z)\,(Zg_n)(z).$$

The next result gives a method for calculating the inverse transform.

Theorem 4.7 *a) Let* $F(z)$ *be a bounded analytic function on* $\{z \in \mathbb{C} : |z| > \rho\}$. *If we define (for* $n \geq 0$*)*

$$f_n = \frac{1}{n!}\, \frac{d^n}{d\xi^n}\Big|_{\xi=0} F(1/\xi)\,,$$

then $(Zf_n)(z) = F(z)$.

b) We also have (for $n \geq 0$)

$$f_n = \frac{1}{2\pi i} \int_\gamma F(z)\, z^{n-1}\, dz\,,$$

where γ is any circumference with center 0 and radius $R > \rho$.

c) As a consequence, if two sequences have the same Z-transform, then both sequences are the same.

Z-TRANSFORMS TABLE

$(a, b, \alpha, \beta \in \mathbb{C})$

$$Z[1](z) = \frac{z}{z-1}\,, \qquad Z[a^n](z) = \frac{z}{z-a}\,,$$

$$Z[n](z) = \frac{z}{(z-1)^2}\,, \qquad Z[\,n\,a^n](z) = \frac{az}{(z-a)^2}\,,$$

$$Z[n^2](z) = \frac{z(z+1)}{(z-1)^3}\,, \qquad Z[\,n^2\,a^n](z) = \frac{az(z+a)}{(z-a)^3}\,,$$

$$Z[n^3](z) = \frac{z(z^2+4z+1)}{(z-1)^4}\,, \qquad Z[\,n^3\,a^n](z) = \frac{az(z^2+4az+a^2)}{(z-a)^4}\,,$$

$$Z[\,\cos(\alpha n + \beta)](z) = \frac{z^2\cos\beta - z\cos(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1}\,, \qquad Z[\sin(\alpha n + \beta)](z) = \frac{z^2\sin\beta + z\sin(\alpha-\beta)}{z^2 - 2z\cos\alpha + 1}\,,$$

$$Z\Big[\frac{a^n - b^n}{a-b}\Big](z) = \frac{z}{(z-a)(z-b)} \quad (a \neq b), \qquad Z\Big[\frac{a^{n+1} - b^{n+1}}{a-b}\Big](z) = \frac{z^2}{(z-a)(z-b)} \quad (a \neq b),$$

$$Z[f_n](z) = \frac{1}{z-a}\,, \qquad \text{if } \; f_n = a^{n-1}\,,\; n \geq 1\,,\; f_0 = 0\,.$$