

uc3m

Universidad **Carlos III** de Madrid

Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.1: Complex numbers

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García



1.1. COMPLEX NUMBERS

1.1. Compute the real and imaginary parts of

$$a) w = \frac{1}{i} + \frac{1}{1+i}, \quad b) w = \frac{1}{(3+2i)^2}, \quad c) w = \frac{z+1}{z^2+5} \quad \text{if } z = x+iy, \quad x, y \in \mathbf{R}.$$

1.2. Find every complex solution of each equation:

$$a) z^2 = 3 - 4i, \quad b) (z+1)^4 + i = 0.$$

1.3. Compute:

$$\begin{array}{lllll} a) (1+i)^4 & b) (-i)^{-1} & c) \sqrt{1+i} & d) \sqrt{1+\sqrt{i}} & e) \sqrt{\sqrt{-i}} \\ f) (1+i)^n + (1-i)^n & g) \sqrt{1-i\sqrt{3}} & h) i^{1/4} & i) (-i)^{1/4}. & \end{array}$$

1.4. Prove that the roots of the equation $z^n = a$, with $a \neq 0$, are the vertices of a regular polygon.

1.5. Show that every complex numbers z, w satisfy

$$|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2).$$

Hint: $|a|^2 = a\bar{a}$ for every $a \in \mathbf{C}$.

1.6. Compute the moduli of the following complex numbers:

$$a) z = i(2+3i)(5-2i)(-2-i)^{-1}, \quad b) w = \frac{(2-3i)^2}{(8+6i)^2}.$$

Hint: $|ab| = |a||b|$, $|a/b| = |a|/|b|$ and $|a^n| = |a|^n$.

1.7. Let w be a n -th root of the unity, i.e., $w^n = 1$. Show that if $w \neq 1$, then:

$$a) 1 + w + w^2 + \dots + w^{n-1} = 0, \quad b) 1 + 2w + 3w^2 + \dots + nw^{n-1} = \frac{n}{w-1}.$$

What are the values if $w = 1$?

Hints: a) It is the sum of a geometric progression. b) It is the sum of an arithmetic-geometric progression.

1.8. a) Prove Lagrange's trigonometric identities:

$$\begin{aligned} 1 + \cos \theta + \cos 2\theta + \dots + \cos n\theta &= \frac{1}{2} \left(1 + \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)} \right), \\ \sin \theta + \sin 2\theta + \dots + \sin n\theta &= \frac{\sin((n+1)\theta/2) \sin(n\theta/2)}{\sin(\theta/2)}, \end{aligned}$$

where θ is an angle such that $\sin(\theta/2) \neq 0$.

Hint: Use the formula for the sum of a geometric progression

$$1 + w + w^2 + \dots + w^n = \frac{w^{n+1} - 1}{w - 1}, \quad \text{with } w = e^{i\theta}.$$

b) Compute the sums:

$$\begin{aligned} & \cos(\alpha + \theta) + \cos(\alpha + 2\theta) + \cdots + \cos(\alpha + n\theta), \\ & \sin(\alpha + \theta) + \sin(\alpha + 2\theta) + \cdots + \sin(\alpha + n\theta). \end{aligned}$$

Hint: Use the formulas in the previous item.

1.9. Describe the set of complex numbers satisfying:

$$a) \operatorname{Im}(z + 5) = 0, \quad b) \operatorname{Re}(z^2 + 5) = 0.$$

1.10 Let z_1, z_2, \dots, z_n be complex numbers. Prove that

$$\left| \sum_{j=1}^n z_j \right| \leq \sum_{j=1}^n |z_j|.$$

1.11. If $|z| = 1$, prove that we have, for every complex numbers a, b ,

$$\left| \frac{az + b}{\bar{b}z + \bar{a}} \right| = 1.$$

Hint: Use that $|az + b| = |\bar{z}(az + b)|$ (since $|\bar{z}| = |z| = 1$) and $|\bar{w}| = |w|$ for every $w \in \mathbf{C}$.

1.12. Consider the complex number $z = (1 + i\sqrt{3})/(1 - i\sqrt{3})$.

- Compute its binomial and exponential forms.
- Prove that $z^4 = z$.
- Compute the other fourth roots of z .
- Compute every complex number which is fourth root of itself.

1.13. Find the locus of the points z of the complex plane verifying $|z - z_1| = |z - z_2|$, if z_1 and z_2 are two complex numbers.

1.14. Let z_1, z_2, z_3 be three vertices of a parallelogram. Find its fourth vertex z_4 opposite to z_2 .

Hint: Use the geometric interpretation of the sum of complex numbers.

1.15. Let z_1, z_2 be two vertices of an equilateral triangle. Find its third vertex z_3 .

Hint: Use the geometric interpretation of the product of complex numbers.