uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.2: Holomorphic functions

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1.2. HOLOMORPHIC FUNCTIONS

2.1. At what points are the following functions differentiable in complex sense?

$$
\begin{aligned}
&a)\ f(z) = |z|\,,\\
&b)\ f(z) = |z|^2\,,\\
&c)\ f(z) = |z|^\alpha\,, \quad \alpha > 0\,,\\
&d)\ f(z) = \sqrt{|xy|}\,, \quad z = x + iy\,,\\
&e)\ f(z) = \overline{z}\,,\\
&f)\ f(z) = h(x)\,, \quad h \in C^1(\mathbf{R}, \mathbf{C})\,,\\
&g)\ f(z) = h(y)\,, \quad h \in C^1(\mathbf{R}, \mathbf{C})\,,\\
&h)\ f(z) = h(x, y)\,, \quad h \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})\,,\\
&i)\ f(z) = \frac{1}{z^2} + |z|^2\,,\\
&j)\ f(z) = z \operatorname{Re} z\,,\\
&k)\ f(z) = \frac{1}{(z + 1/z)^2}\,,\\
&l)\ f(z) = e^z := e^x \cos y + ie^x \sin y\,,\\
&m)\ f(z) = e^{z^3/(z+3)}.
\end{aligned}
$$

Solutions: $a)$ $\emptyset$, $b)$ $\{0\}$, $c)$ $\emptyset$ if $0 < \alpha \leq 1$, $\{0\}$ if $\alpha > 1$, $d)$ $\emptyset$, $e)$ $\emptyset$, $f)$ $\{z = x + iy : h'(x) = 0, y \in \mathbf{R}\}$, $g)$ $\{z = x + iy : h'(y) = 0, x \in \mathbf{R}\}$, $h)$ $\{z = x + iy \in \mathbf{C} : \partial h(x, y)/\partial x = 0 = \partial h(x, y)/\partial y\}$, $i)$ $\emptyset$, $j)$ $\{0\}$, $k)$ $\mathbf{C} \setminus \{0, i, -i\}$, $l)$ $\mathbf{C}$, $m)$ $\mathbf{C} \setminus \{-3\}$.

2.2. Find a holomorphic function f with real part:

$$
\begin{aligned}
&a)\ u(x, y) = ax + by + c\,, \qquad a, b, c \in \mathbf{R}\,,\\
&b)\ u(x, y) = e^{-x}(x \sin y - y \cos y)\,,\\
&c)\ u(x, y) = e^{-y} \cos x\,,\\
&d)\ u(x, y) = \frac{x}{x^2 + y^2}\,,\\
&e)\ u(x, y) = e^{x^2 - y^2} \sin 2xy\,,\\
&f)\ u(x, y) = x\, e^{-x} \cos y + y\, e^{-x} \sin y\,,\\
&g)\ u(x, y) = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy\,,\\
&h)\ u(x, y) = \log \sqrt{x^2 + y^2}\,,\\
&i)\ u(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}\,,\\
&j)\ u(x, y) = \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} x^{2n-2k} y^{2k}\,.
\end{aligned}
$$

In which domain is f holomorphic in each case?

Solutions: The following functions are harmonic conjugates: $a)$ $v(x, y) = ay - bx$, $b)$ $v(x, y) = e^{-x}(x \cos y + y \sin y)$, $c)$ $v(x, y) = e^{-y} \sin x$, $d)$ $v(x, y) = -y/(x^2 + y^2)$, $e)$ $v(x, y) = -e^{x^2 - y^2} \cos 2xy$, $f)$ $v(x, y) = e^{-x}(y \cos y - x \sin y)$, $g)$ $v(x, y) = \cos x \sinh y - 2 \sin x \cosh y + 2xy - 2(x^2 - y^2)$, $h)$ $v(x, y) = \arctan(y/x)$, $i)$ $v(x, y) = -2xy/(x^2 + y^2)^2$, $j)$ $v(x, y) = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{2k+1} x^{2n-2k-1} y^{2k+1}$.

2.3. a) Prove that a harmonic function u on an open connected set U has a harmonic conjugate if and only if there exists a holomorphic function f on U such that $f' = u_x - iu_y$.

b) Let $z_0 = x_0 + iy_0$ and $u \in \mathbf{C}^2(D(z_0, r))$ be a harmonic function. Prove that

$$v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\, dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\, dx$$

is a harmonic conjugate of u on $D(z_0, r)$ with $v(x_0, y_0) = c$.

Hint: You can use the Fundamental Theorem of Calculus in order to compute the partial derivatives of $v(x_1, y_1)$.

2.4. Prove that the Cauchy-Riemann equations on polar coordinates are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\,, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}\,.$$

Hint: Use the chain rule.

2.5. Using if required the Cauchy-Riemann equations on polar coordinates, find the holomorphic function verifying:

$a)$ its modulus is $e^{r^2 \cos 2\theta}$,
$b)$ its modulus is $(x^2 + y^2)\, e^x$,
$c)$ its argument is xy,
$d)$ its argument is $\theta + r\,\sin\theta$,
$e)$ its real part is $\dfrac{\cos\theta + \sin\theta}{r}$.

In which domain is f holomorphic in each case?

Hint: $a)$ and $c)$: if $p(z)$ is a polynomial, then $e^{p(z)}$ is a holomorphic function.

Solutions: $a)$ $f(z) = e^{z^2}$, $b)$ $f(z) = z^2 e^z$, $c)$ $f(z) = e^{z^2/2}$, $d)$ $f(z) = ze^z$, $e)$ $\operatorname{Im} f(z) = (\cos\theta - \sin\theta)/r$ and so, $f(z) = (1+i)/z$.

2.6. If $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, with $a, b, c, d \in \mathbf{R}$, for what values of a, b, c, d is u harmonic?

Solution: $c = -3a$, $b = -3d$.

2.7. Find the harmonic functions which can be written as:

$a)$ $g(ax + by)$, $a, b \in \mathbf{R}$, $\quad b)$ $g(x^2 + y^2)$, $\quad c)$ $g(xy)$,
$d)$ $g(x + \sqrt{x^2 + y^2}\,)$, $\quad e)$ $g((x^2 + y^2)/x)$, $\quad f)$ $g(y/x)$,

with g a real function of class C^2. Find in each case the holomorphic functions whose real parts are these functions.

Solutions: If $\alpha, \beta \in \mathbf{R}$, $a)$ $g(t) = \alpha t + \beta$, $b)$ $g(t) = \alpha \log t + \beta$, $c)$ $g(t) = \alpha t + \beta$, $d)$ $g(t) = \alpha\sqrt{t} + \beta$, $e)$ $g(t) = \alpha/t + \beta$, $f)$ $g(t) = \alpha \arctan t + \beta$.

2.8. Find a holomorphic function f on $\mathbf{C}$ such that $f(1+i) = 0$, $f'(0) = 0$ and $\operatorname{Re} f'(z) = 3\,(x^2 - y^2) - 4y$.

Solution: $f(z) = z^3 + 2iz^2 + 6 - 2i$.

2.9. Prove that if u and v are harmonic functions and $\alpha, \beta \in \mathbf{R}$, then $\alpha u + \beta v$ is harmonic. For what harmonic functions u is also harmonic the function u^2?

Hint: $\Delta(u^2) = 2|\nabla u|^2 + 2u\Delta u$.

2.10. Let U be an open connected subset of $\mathbf{C}$ and $f : U \longrightarrow \mathbf{C}$ a holomorphic function. Prove the following statements:

a) If $f'(z) = 0$ on U, then f is constant on U.
b) If $\operatorname{Re} f$ is constant on U, then f is constant on U.
c) If $\operatorname{Im} f$ is constant on U, then f is constant on U.
d) If $|f|$ is constant on U, then f is constant on U.
e) If $\overline{f}$ is holomorphic on U, then f is constant on U.

Hints: Use Cauchy-Riemann equations. d) Recall that 0 is the only vector in the plane which is orthogonal to two linearly independent vectors.

2.11. Assume that f is holomorphic on $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Study if each function g below is holomorphic on $\mathbf{D}$:

$$a)\ g(z) = \overline{f(z)}\,, \qquad b)\ g(z) = f(\overline{z})\,, \qquad c)\ g(z) = \overline{f(\overline{z})}\,, \qquad d)\ g(z) = |f(z)|\,, \qquad e)\ g(z) = f(z)\, f(\overline{z})\,.$$

Do we get the same result if we replace $\mathbf{D}$ by any domain A? And if A satisfies that $z \in A \Leftrightarrow \overline{z} \in A$?

Solution: a), b), d), e) g is holomorphic if and only if f is constant. c) g is holomorphic for every holomorphic function f. The results also hold if A satisfies $z \in A \Leftrightarrow \overline{z} \in A$.

2.12. Prove that if $h : \mathbf{R}^2 \longrightarrow \mathbf{R}$ is of class C^2 and f is holomorphic, then

$$\Delta(h \circ f) = \Delta h(f) \cdot |f'|^2\,.$$

Hint: Use the chain rule.

2.13. Prove that $f(z) = z^2$ is holomorphic on $\mathbf{C}$ by checking that if satisfies the Cauchy-Riemann equations. If $f(z_0) = a + bi$, prove that the curves $\operatorname{Re} f(z) = a$ and $\operatorname{Im} f(z) = b$ are orthogonal at z_0.

Hint: Prove that the inner product of the tangent vectors of the two curves is 0.

2.14. Find all the holomorphic functions which can be written as $u(x) + i\,v(y)$.

Solution: $f(z) = \alpha z + \beta$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{C}$.

2.15. a) If $Q(z)$ is a polynomial with different roots $z_1, \ldots, z_n$, and if $P(z)$ is a polynomial of degree less than n, prove that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(z_k)}{Q'(z_k)\,(z - z_k)}\,.$$

b) Prove that there exists a unique polynomial p of degree less than n with $p(z_k) = w_k$ (*Lagrange's interpolation formula*).

Hints: a) $P(z)$ and $\sum_{k=1}^{n} \frac{P(z_k)Q(z)}{Q'(z_k)\,(z-z_k)}$ have the same values at $z_1, \ldots, z_n$. b) Use the formula in the previous item.