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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.3: Power series

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1.3. POWER SERIES

3.1. Compute the radius of convergence of the following power series:

$$a)\ \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\,z^n\,,\qquad b)\ \sum_{n=1}^{\infty}\frac{n!}{n^n}\,z^n\,,\qquad c)\ \sum_{n=0}^{\infty}2^{-n}\,z^n\,,$$

$$d)\ \sum_{n=0}^{\infty}2^n\,z^n\,,\qquad e)\ \sum_{n=1}^{\infty}n^n\,z^n\,,\qquad f)\ \sum_{n=1}^{\infty}a^n\,z^n\quad(a\in\mathbf{C})\,,$$

$$g)\ \sum_{n=0}^{\infty}(n+a^n)\,z^n\,,\quad(a\in\mathbf{C})\,,\qquad h)\ \sum_{n=0}^{\infty}z^{n!}\,,\qquad i)\ \sum_{n=0}^{\infty}a^{n^2}\,z^n\quad(a\in\mathbf{C})\,,$$

$$j)\ \sum_{n=1}^{\infty}\exp\left(\frac{n^n\sqrt{2\pi n}}{e^n}\right)z^n\,,\qquad k)\ \sum_{n=1}^{\infty}n^4\,z^n\,,\qquad l)\ \sum_{n=1}^{\infty}n^{\alpha}\,z^n\quad(\alpha\in\mathbf{R})\,,$$

$$m)\ \sum_{n=0}^{\infty}z^{2^n}\,,\qquad n)\ \sum_{n=0}^{\infty}\cos(in)\,z^n\,,\qquad o)\ \sum_{n=1}^{\infty}a^{n^2}\,z^{1+2+\cdots+n}\,,$$

$$p)\ \sum_{n=0}^{\infty}(\cos a_n+i\,\sin a_n)\,z^n\,,\quad\{a_n\}\subset\mathbf{R}\,,\qquad q)\ \sum_{n=0}^{\infty}\big(3+(-1)^n\big)^n\,z^n\,,$$

$$r)\ 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}\,z^n\,,\qquad\alpha,\beta,\gamma\in\mathbf{C},\ \gamma\notin\mathbf{Z}\,.$$

Solutions: $a)$ $R=1/4$, $b)$ $R=e$, $c)$ $R=2$, $d)$ $R=1/2$, $e)$ $R=0$, $f)$ $R=1/|a|$, $g)$ $R=1/|a|$ if $|a|>1$, $R=1$ if $|a|\le 1$, $h)$ $R=1$, $i)$ $R=\infty$ if $|a|<1$, $R=1$ if $|a|=1$, $R=0$ if $|a|>1$, $j)$ $R=0$, $k)$ $R=1$, $l)$ $R=1$, $m)$ $R=1$, $n)$ $R=1/e$, $o)$ $R=1/|a|^2$, $p)$ $R=1$, $q)$ $R=1/4$, $r)$ $R=1$.

3.2. If the radius of convergence of the series $\sum_{n=0}^{\infty}c_nz^n$ is R, find the radius de convergence of $\sum_{n=0}^{\infty}a_nz^n$ where a_n is:

$$a)\ a_n=n^kc_n\,,\qquad b)\ a_n=(2^n-1)\,c_n\,,\qquad c)\ a_n=\frac{c_n}{n!}\,,\ \text{if } R>0\,,$$

$$d)\ a_n=n^nc_n\,,\ \text{if } R<\infty\,,\qquad e)\ a_n=c_n^k\,,\qquad f)\ a_n=(1+z_0^n)\,c_n\,,\ \text{with } |z_0|\neq 1\,.$$

Solutions: $a)$ R, $b)$ $R/2$, $c)$ ∞, $d)$ 0, $e)$ R^k, $f)$ R if $|z_0|<1$, and $R/|z_0|$ if $|z_0|>1$.

3.3. If the radius of convergence of the series $\sum_{n=0}^{\infty}c_nz^n$ is R $(0\le R\le\infty)$, find the radius of convergence ρ of:

$$a)\ \sum_{n=0}^{\infty}c_{2n}z^n\,,\qquad b)\ \sum_{n=0}^{\infty}c_{kn}z^n\,,\qquad c)\ \sum_{n=0}^{\infty}c_nz^{2n}\,,\qquad d)\ \sum_{n=0}^{\infty}c_nz^{kn}\,.$$

Solutions: $a)$ $\rho\ge R^2$, $b)$ $\rho\ge R^k$, $c)$ $\rho=R^{1/2}$, $d)$ $\rho=R^{1/k}$.

3.4. If the radii of convergence of the series $\sum_{n=0}^{\infty}a_nz^n$, $\sum_{n=0}^{\infty}b_nz^n$ are R_1, R_2, respectively, find the radius of convergence of:

$$a)\ \sum_{n=0}^{\infty}(a_n+b_n)\,z^n\,,\qquad b)\ \sum_{n=0}^{\infty}a_nb_nz^n\,,\qquad c)\ \sum_{n=0}^{\infty}\frac{a_n}{b_n}\,z^n\quad(\text{if } b_n\neq 0)\,.$$

Solutions: $a)$ $R\ge\min\{R_1,R_2\}$; if $R_1\neq R_2$, then $R=\min\{R_1,R_2\}$; $b)$ $R\ge R_1R_2$; if there exists the limit of $|a_n|^{1/n}$ or $|b_n|^{1/n}$, then $R=R_1R_2$; $c)$ if the limit of $|b_n|^{1/n}$ does not exist, can occur $R=0$; if there exists the limit of $|b_n|^{1/n}$, then $R=R_1/R_2$.

3.5. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, z^{n(n+1)}$$

and study its convergence for $z = 1, -1, i$.
Solution: $R = 1$ and the series converges at the three points.

3.6. Sum, for appropriate values of z, the series

$$a)\ \sum_{n=1}^{\infty} n z^n\,, \qquad b)\ \sum_{n=1}^{\infty} \frac{z^n}{n}\,, \qquad c)\ \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}\,,$$
$$d)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\, z^n\,, \qquad e)\ \sum_{n=0}^{\infty} \binom{n}{2} z^n\,.$$

Solutions: *a)* $z/(1-z)^2$, *b)* $-\log(1-z)$, *c)* $\log\sqrt{(1+z)/(1-z)}$, *d)* $\log(1+z)$, *e)* $z^2/(1-z)^3$.

3.7. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, write the series $\sum_{n=0}^{\infty} n^3 a_n z^n$ in terms of f and its derivatives, in its disk of convergence.
Solution: $\sum_{n=0}^{\infty} n^3 a_n z^n = z\, f'(z) + 3z^2 f''(z) + z^3 f'''(z)$.

3.8. a) Consider the power series $\sum_{n=0}^{\infty} a_n z^n$; if there exists the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R}\,, \qquad \text{with } 0 < R < \infty,$$

prove that, in fact, R is the radius of convergence of the series.

Hint: If $|z| \le r < R$, prove that the series absolutely and uniformly converges. If $|z| = \rho > R$, prove that the sequence $\{|a_n z^n|\}$ is not bounded.

b) Consider the power series $\sum_{n=1}^{\infty} a_n z^n$, with

$$a_n = \begin{cases} \dfrac{1}{n^2}\,, & \text{if } n \text{ is even,} \\ \dfrac{1}{n^3}\,, & \text{if } n \text{ is odd.} \end{cases}$$

Compute the inverse of the radius of convergence. Is it equal to the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$? Does this contradict the result in item a)?
Solution: b) There is no contradiction since the limit of the quotient does not exist.

3.9. Write the following functions as power series and compute their radii of convergence:

a) $(1-z)^{-m}$ (m is a positive integer) as power series about 0.

b) $\dfrac{2z+3}{z+1}$ and $\dfrac{2z+3}{(z+1)^2}$ as power series about 1.

c) $\dfrac{1}{4+z^2}$, $\dfrac{1}{z^2-5z+6}$, $\dfrac{z}{(z-1)^2}$ as power series about 0.

d) $\dfrac{1}{az+b}$, with $a, b \in \mathbf{C}$ and $b \neq 0$, as power series about 0.

e) $\dfrac{6z}{z^2-4z+13}$, as power series about 0. *Hint*: Use the previous item.

f) $\sin^2 z$, as power series about 0.

g) $\dfrac{1}{(z+1)^2}$ as power series about 0.

h) $\dfrac{z^2}{(z+1)^2}$ as power series about 0.

i) $\log\dfrac{1+z}{1-z} = \displaystyle\int_0^z \frac{dw}{1+w} + \int_0^z \frac{dw}{1-w}$ as power series about 0.

j) $\displaystyle\int_0^z e^{w^2}\,dw$ as power series about 0.

k) $\displaystyle\int_0^z \frac{\sin w}{w}\,dw$ as power series about 0.

l) $\arctan z = \displaystyle\int_0^z \frac{1}{1+w^2}\,dw$ as power series about 0.

Solutions: *a)* $\sum_{n=0}^{\infty}\binom{m+n-1}{m-1}z^n$, $R=1$,
b) $5/2+\sum_{n=1}^{\infty}(-1)^n(z-1)^n/2^{n+1}$, $R=2$, $\sum_{n=0}^{\infty}(-1)^n(n+5)(z-1)^n/2^{n+2}$, $R=2$,
c) $\sum_{n=0}^{\infty}(-1)^n z^{2n}/4^{n+1}$, $R=2$, $\sum_{n=0}^{\infty}(2^{-n-1}-3^{-n-1})z^n$, $R=2$, $\sum_{n=0}^{\infty} nz^n$, $R=1$,
d) $b^{-1}\sum_{n=0}^{\infty}(-a/b)^n z^n$, $R=|b/a|$,
e) $i\sum_{n=0}^{\infty}((2+3i)^{-n}-(2-3i)^{-n})z^n$, $R=\sqrt{13}$,
f) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{2n-1}z^{2n}/(2n)!$, $R=\infty$,
g) $\sum_{n=0}^{\infty}(-1)^n(n+1)z^n$, $R=1$,
h) $\sum_{n=2}^{\infty}(-1)^n(n-1)z^n$, $R=1$,
i) $2\sum_{n=0}^{\infty} z^{2n+1}/(2n+1)$, $R=1$,
j) $\sum_{n=0}^{\infty} z^{2n+1}/((2n+1)n!)$, $R=\infty$,
k) $\sum_{n=0}^{\infty}(-1)^n z^{2n+1}/((2n+1)(2n+1)!)$, $R=\infty$,
l) $\sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)$, $R=1$.

3.10. Find an analytic function $f(z)$ on $\mathbf{C}$ such that $f^{(n)}(-i)=(-i)^n$, for every $n\in\mathbf{N}$.
Solution: $f(z)=\sum_{n=0}^{\infty}(-i)^n(z+i)^n/n! = e^{1-iz}$.

3.11. Assume that the power series $\sum_{n=0}^{\infty} a_n(z-2)^n$ converges at $z=0$. Can it diverge at $z=3$?
Solution: The power series converges on $\{|z-2|<2\}$.

3.12. Assume that the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n$ is $R=1$, and that the coefficients c_n satisfy $c_0\ge c_1\ge\cdots$, and $\lim_{n\to\infty} c_n=0$. Prove that the series $\sum_{n=0}^{\infty} c_n z^n$ converges on $|z|=1$ except perhaps at $z=1$.

Hint: You can use the following *Dirichlet criterion*: If $\{b_n\}\subset\mathbf{C}$ and its partial sums are a bounded sequence, $c_0\ge c_1\ge\cdots$, and $\lim_{n\to\infty} c_n=0$, then the series $\sum_{n=0}^{\infty} b_n c_n$ is convergent.

3.13. Study the convergence of the following power series on the boundary of their disks of convergence:

$$a)\ \sum_{n=1}^{\infty} z^n\,,\qquad b)\ \sum_{n=1}^{\infty}\frac{z^n}{n}\,,\qquad c)\ \sum_{n=1}^{\infty}\frac{z^n}{n^2}\,,\qquad d)\ \sum_{n=1}^{\infty}\frac{z^{n!}}{n^2}\,,$$

$$e)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,z^n\,,\qquad f)\ \sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}\,z^{3n-1}\,,\qquad g)\ \sum_{n=1}^{\infty}\frac{z^{pn}}{n}\,,\quad (p\in\mathbf{N})\,,$$

$$h)\ \sum_{n=1}^{\infty}\frac{z^n}{n^{\alpha}}\quad(\alpha\in\mathbf{R})\,,\qquad i)\ \sum_{n=0}^{\infty}(e^n+e^{-n})\,z^n\,.$$

Solutions: *a)* diverges at every point in $\{|z|=1\}$; *b)* converges at every point in $\{|z|=1\}$ except at $z=1$; *c), d)* converges at every point in $\{|z|=1\}$; *e)* converges at every point in $\{|z|=1\}$ except at $z=-1$;

$f)$ converges at every point in $\{|z| = 1\}$ except if $z^3 = -1$; $g)$ converges at every point in $\{|z| = 1\}$ except if $z^p = 1$; $h)$ diverges at every point in $\{|z| = 1\}$ if $\alpha \le 0$, converges at every point in $\{|z| = 1\}$, except at $z = 1$, if $0 < \alpha \le 1$, and converges at every point in $\{|z| = 1\}$ if $\alpha > 1$; $i)$ diverges at every point in $\{|z| = 1/e\}$.

3.14. For which values of z are convergent the following series?

$$a)\ \sum_{n=1}^{\infty} \left(\frac{z}{1+z}\right)^n, \qquad b)\ \sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}\,.$$

$c)$ What is the value of the series $a)$ at the points where it converges?

Hint for b): Consider the cases $|z| < 1$ and $|z| > 1$ separately.

Solutions: $a)$ $\{\mathrm{Re}\, z > -1/2\}$, $b)$ $\{|z| \ne 1\}$, $c)$ z.