uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.4: Elementary functions

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

1.4. ELEMENTARY FUNCTIONS

4.1. Write on exponential form the following complex numbers: 1, -1, i, $-i$, $1+i$, $1-i$, $-1+i$, $-1-i$.
Solutions: $e^{i\cdot 0}$, $e^{i\pi}$, $e^{i\pi/2}$, $e^{-i\pi/2}$, $\sqrt{2}\,e^{i\pi/4}$, $\sqrt{2}\,e^{-i\pi/4}$, $\sqrt{2}\,e^{i3\pi/4}$, $\sqrt{2}\,e^{-i3\pi/4}$.

4.2. Find the moduli and the principal values of the arguments (obtained by taking $\arg z \in (-\pi,\pi]$) of the following complex numbers: e^{2+i}, e^{2-3i}, e^{3+4i}, e^{-3-4i}, $-ae^{i\theta}$ ($a>0$, $|\theta|\le\pi$), $e^{-i\theta}$ ($|\theta|\le\pi$), $e^{i\alpha}-e^{i\beta}$ ($0\le\beta<\alpha\le 2\pi$).
Solutions: $|e^{2+i}| = e^2$, $\arg(e^{2+i}) = 1$.
$|e^{2-3i}| = e^2$, $\arg(e^{2-3i}) = -3$.
$|e^{3+4i}| = e^3$, $\arg(e^{3+4i}) = 4-2\pi$.
$|e^{-3-4i}| = e^{-3}$, $\arg(e^{-3-4i}) = 2\pi-4$.
$|-ae^{i\theta}| = a$, $\arg(-ae^{i\theta}) = \pi+\theta$ if $\theta\in[-\pi,0]$, $\arg(-ae^{i\theta}) = \theta-\pi$ if $\theta\in(0,\pi]$.
$|e^{-i\theta}| = 1$, $\arg(e^{-i\theta}) = -\theta$ if $\theta\in[-\pi,\pi)$, $\arg(e^{-i\theta}) = \pi$ if $\theta=\pi$.

4.3. Find the values of: $\log(-e)$, 1^i, $\sin i$, $\cos i$, $\cos(2+i)$, $\tan(1+i)$, $\mathrm{cotan}\,(\pi/4 - i\log 2)$, $\mathrm{cotanh}\,(2+i)$, $\tanh(\log 3 + \pi i/4)$.
Solutions: $\log(-e) = \{1+(2k+1)\pi i : \ k\in\mathbf{Z}\}$. $1^i = \{e^{-2k\pi} : \ k\in\mathbf{Z}\}$. $\cos i = (e+1/e)/2 = \cosh 1$. $\tanh(\log 3+\pi i/4) = \frac{40}{41}+\frac{9}{41}\,i$.

4.4. Let a be a complex number different from 0 and r a real number. Compute the set of values of a^r in the following cases:

a) r integer, b) $r=m/n$ rational, c) r irrational.

Solutions: $a^r = e^{r\log a} = \{w\in\mathbf{C}: \ w = |a|^r e^{ir\arg a} e^{2kr\pi i},\ k\in\mathbf{Z}\}$. Hence: a) If r is integer, then $e^{2kr\pi i} = 1$ for every $k\in\mathbf{Z}$, and a^r has a unique value $a^r = |a|^r e^{i\arg a}$. b) If $r=m/n\in\mathbf{Q}$, then $r = m_1+m_2/n$ with $m_1,m_2\in\mathbf{Z}$, $0\le m_2<n$ and $gcd(m_2,n)=1$. Thus, $e^{2kr\pi i} = e^{2km_2\pi i/n}$ has n different values for $m_2 = 0,1,\ldots,n-1$. Hence, a^r has n values. c) If r is irrational, then a^r has infinite values.

4.5. Compute the values of $(2^i)^2$, $(2^2)^i$ and 2^{2i}.
Solutions: $(2^i)^2 = \{e^{2(k'+k'')\pi}(\cos(\log 2)+i\sin(\log 2))^2,\ k',k''\in\mathbf{Z}\} = \{e^{2k\pi}(\cos(\log 4)+i\sin(\log 4)),\ k\in\mathbf{Z}\}$.
$(2^2)^i = 4^i = \{e^{2k\pi}(\cos(\log 4)+i\sin(\log 4)),\ k\in\mathbf{Z}\}$.
$2^{2i} = \{e^{4k\pi}(\cos(\log 4)+i\sin(\log 4)),\ k\in\mathbf{Z}\}$.

4.6. Find the values of the following powers: $1^{\sqrt{2}}$, $(-2)^{\sqrt{2}}$, 2^i, 1^{-i}, i^i, $(1-i)^{1+i}$, $(3-4i)^{1+i}$.
Solutions: $1^{\sqrt{2}} = \{w = e^{2\sqrt{2}k\pi i},\ k\in\mathbf{Z}\}$.
$(-2)^{\sqrt{2}} = \{w\in\mathbf{C}: \ w = e^{\sqrt{2}\log 2}e^{\sqrt{2}(2k-1)\pi i},\ k\in\mathbf{Z}\}$.
$2^i = \{w = e^{-2k\pi}e^{i\log 2},\ k\in\mathbf{Z}\}$.
$1^{-i} = \{w\in\mathbf{C}: \ w = e^{2k\pi},\ k\in\mathbf{Z}\}$.
$i^i = \{e^{-(2k+1/2)\pi},\ k\in\mathbf{Z}\}$.
$(1-i)^{1+i} = \{\sqrt{2}\,e^{-(2k-1/4)\pi}e^{(\log\sqrt{2}-\pi/4)i},\ k\in\mathbf{Z}\}$.
$(3-4i)^{1+i} = \{5\,e^{\arctan 4/3-2k\pi}e^{(\log 5-\arctan 4/3)i},\ k\in\mathbf{Z}\}$.

4.7. Prove that the following formulas hold for every complex numbers z,w:

a) $\sin(z+w) = \sin z\cos w + \cos z\,\sin w$.
b) $\tan(z+w) = \dfrac{\tan z+\tan w}{1-\tan z\tan w}$.
c) $\cosh(z+w) = \cosh z\cosh w + \sinh z\,\sinh w$.
d) $\tanh(z+w) = \dfrac{\tanh z+\tanh w}{1+\tanh z\tanh w}$.
e) $\sin(iz) = i\,\sinh z$, $\cos(iz) = \cosh z$, $\tan(iz) = i\tanh z$.
f) $\cos^2 z+\sin^2 z = 1$, $\cosh^2 z-\sinh^2 z = 1$.

Hint: Write these functions in terms of exponentials.

4.8. Compute the inverses of the trigonometric and hyperbolic functions in terms of the logarithm. Compute also their derivatives.
Solutions:

$$\begin{aligned}
\operatorname{arsinh} z &= \log\left(z+\sqrt{z^2+1}\right), & \operatorname{arcosh} z &= \log\left(z+\sqrt{z^2-1}\right), \\
\operatorname{artanh} z &= \frac{1}{2}\log\frac{1+z}{1-z}, & \operatorname{arcotanh} z &= \frac{1}{2}\log\frac{z+1}{z-1}, \\
\arcsin z &= -i\log\left(iz+\sqrt{1-z^2}\right), & \arccos z &= -i\log\left(z+\sqrt{z^2-1}\right), \\
\arctan z &= \frac{-i}{2}\log\frac{i-z}{i+z}, & \operatorname{arccotan} z &= \frac{-i}{2}\log\frac{z+i}{z-i}.
\end{aligned}$$

4.9. Find every complex number satisfying each of the following equations:

a) $e^{4z}=i$, b) $\sin z=3$, c) $\sin z=i$, d) $\operatorname{cotan} z=i+1$, e) $\sin z=0$, f) $\cos z=0$, g) $\tan z=0$, h) $e^{e^z}=1$, i) $\sin z+\cos z=2$, j) $\sin z-\cos z=i$.

Solutions: In what follows, $k,k'\in\mathbf{Z}$, $k'\neq 0$. a) $z=(\pi/2+2k\pi)i/4$, b) $z=\pi/2+2k\pi-i\log(3\pm 2\sqrt{2})$, c) $z=2k\pi-i\log(\sqrt{2}-1)$, $z=(2k+1)\pi-i\log(\sqrt{2}+1)$, d) $z=\frac{1}{2}\arctan 2+k\pi-\frac{1}{4}i\log 5$, e) $z=k\pi$, f) $z=\pi/2+k\pi$, g) $z=k\pi$, h) $z=\log(2k'\pi)+i(\pi/2+2k\pi)$ if $k'>0$, $z=\log(2k'\pi)+i(-\pi/2+2k\pi)$ if $k'<0$.

4.10. a) Prove with an example that $\log(zw)\neq\log z+\log w$, where log denotes the principal value of the logarithm (obtained by taking $\arg z\in(-\pi,\pi]$).

b) Prove that the formula $\log(zw)=\log z+\log w$ holds if $\log z$ denotes the set of every logarithm of z.

Solution: a) Consider $z=w=-1$.