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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.5: Complex integration

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1.5. COMPLEX INTEGRATION

5.1. Compute the following integrals:

a) $\displaystyle\int_\gamma x\,dz$, where γ is the segment joining 0 with $1+i$.

b) $\displaystyle\int_\gamma x\,dz$, where γ is the circumference with radius r centered at 0.

c) $\displaystyle\int_\gamma \frac{dz}{z}$, with $\gamma(t) = \cos t + 2i\sin t,\ t \in [0, 2\pi]$.

d) $\displaystyle\int_\gamma \frac{dz}{z^2}$, with $\gamma(t) = \cos t + 2i\sin t,\ t \in [0, 2\pi]$.

e) $\displaystyle\int_\gamma \frac{dz}{z^2-1}$, with $\gamma(t) = 1 + e^{it},\ t \in [0, 2\pi]$.

f) $\displaystyle\int_\gamma |z|\,\overline{z}\,dz$, where γ is the boundary of the half-disk $\{z \in \mathbf{C} : |z| \le 1, \operatorname{Im} z \ge 0\}$.

g) $\displaystyle\int_\gamma |z-1|\,|dz|$, where γ is the unit circumference (the boundary of the disk with radius 1 centered at 0).

h) $\displaystyle\int_\gamma |z|^2\,dz$, where γ is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

i) $\displaystyle\int_\gamma \frac{|z|}{|1-z|^2}\,|dz|$, where γ is the circumference with radius r $(0 < r < 1)$ centered at 0.

j) $\displaystyle\int_\gamma \frac{dz}{|z-a|^2}$, where γ is the circumference with radius r centered at 0, and $a \in \mathbf{C}$ with $|a| \ne r$.

Hints: i) prove first that if $0 \le r < R$, then

$$\frac{1}{R^2 - 2rR\cos t + r^2} = \frac{1}{R^2 - r^2}\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos nt\right).$$

j) Recall that

$$|z-a|^2 = (z-a)(\bar z - \bar a) = (z-a)(r^2/z - \bar a).$$

Solutions: a) $(1+i)/2$, b) $\pi r^2 i$, c) $2\pi i$, d) 0, e) πi, f) πi, g) 8, h) $(i-1)/3$, i) $2\pi r^2/(1-r^2)$, j) $2\pi i a/(r^2 - |a|^2)$, if $|a| < r$, $2\pi i a r^2/[|a|^2(|a|^2 - r^2)]$, if $|a| > r$.

5.2. Compute $\displaystyle\int_\gamma \overline{z}\,dz$, if γ is:

a) $\gamma(t) = t^2 + it$, with $0 \le t \le 2$,
b) the polygonal joining the points 0 with $2i$, and $2i$ with $4+2i$.

Can an holomorphic function exist with derivative $\overline{z}$?

Solutions: a) $10 - 8i/3$. b) $10 - 8i$. No.

5.3. Compute $\displaystyle\int_\gamma \frac{dz}{\overline{z}^2}$, if γ is:

a) $\gamma(t) = e^{i(\pi - t)}$, with $0 \le t \le \pi$,
b) $\gamma(t) = e^{it}$, with $\pi \le t \le 2\pi$.

Do the answers of a) and b) imply that there exists an holomorphic function f with $f'(z) = 1/\overline{z}^2$?

Solutions: a) 2/3. b) 2/3. No.

5.4. a) Compute for each $n \in \mathbf{Z}$ and $a \in \mathbf{C}$ the integral:

$$I = \int_\gamma (z-a)^n \, dz,$$

where γ is any circumference with $a \notin \gamma$.

b) Prove that there is no holomorphic function f on $\mathbf{C}\backslash\{0\}$ such that $f'(z) = 1/z$.

c) If $\gamma = \{z \in \mathbf{C} : |z| = 2\}$, compute the following integral (by using the index):

$$\int_\gamma \frac{dz}{z^2-1} \, .$$

d) If $\gamma = \{z \in \mathbf{C} : |z| = 3\}$, compute the following integral:

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32} \, dz \, .$$

Solutions: a) $I = 2\pi i$ if $n = -1$ and a is in the interior of the circumference, and $I = 0$ otherwise; c) 0; d) $4\pi i$.

5.5. a) If $f : [a,b] \longrightarrow \mathbf{C}$ is an integrable function, prove that:

$$\Big| \int_a^b f(t) \, dt \Big| \leq \int_a^b |f(t)| \, dt$$

Hint: If $\int_a^b f(t) \, dt = R \, e^{i\alpha}$, then $R = \int_a^b \mathrm{Re}\,(e^{-i\alpha} f(t)) \, dt$.

b) Let $D \subset \mathbf{C}$ be an open set, $f : D \longrightarrow \mathbf{C}$ a continuous function and γ a curve contained on D. Prove that:

$$\Big| \int_\gamma f(z) \, dz \Big| \leq \int_\gamma |f(z)| \, |dz| \, .$$

c) With the hypotheses in the previous item, prove that

$$\Big| \int_\gamma f(z) \, dz \Big| \leq M \, \mathrm{length} \, \gamma \, ,$$

if $|f(z)| \leq M$ on γ.

d) Prove that if $\gamma(t) = e^{it}$ with$t \in [0, \pi]$, we have

$$\Big| \int_\gamma \frac{e^z}{z} \, dz \Big| \leq \pi \, e \, .$$

e) Prove that if γ is the unit circumference positively oriented, we have

$$\Big| \int_\gamma \frac{\sin z}{z^2} \, dz \Big| \leq 2\pi \, e \, .$$

It would have the same inequality if γ were oriented in the opposite direction?

Solution: e) Yes.

5.6. a) If $f_n \to f$ uniformly on $\gamma([a,b])$, prove that $\displaystyle\int_\gamma f_n \to \int_\gamma f$ as $n \to \infty$.

Hint: Use item c) in the previous exercise.

b) Consider the curves $\gamma : [0, 2\pi] \longrightarrow \mathbf{C}$ with $\gamma(t) = re^{it}$ and $\gamma_n : [0, 2\pi] \longrightarrow \mathbf{C}$ with $\gamma_n(t) = (1 - 1/n)re^{it}$. If f is a continuous function on the closed disk centered at 0 with radius r, prove that:

$$\int_\gamma f(z) \, dz = \lim_{n\to\infty} \int_{\gamma_n} f(z) \, dz \, .$$

Hint: You can use item a).