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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.6: Cauchy's integral formula

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1.6. CAUCHY'S INTEGRAL FORMULA

6.1. Compute the following integrals:

$$a)\ \int_{|z|=1} \frac{\cos z}{z}\, dz\,, \qquad b)\ \int_{|z|=1} \frac{\sin z}{z^2}\, dz\,, \qquad c)\ \int_{|z|=3} \frac{e^z + z}{z-2}\, dz\,,$$

$$d)\ \int_{|z|=2} \frac{z^2}{z-1}\, dz\,, \qquad e)\ \int_{|z|=2} \frac{z^2-1}{z^2+1}\, dz\,, \qquad f)\ \int_{|z|=2} \frac{dz}{z^2+2z-3}\,,$$

$$g)\ \int_{|z|=2} \frac{|z|\, e^z}{z^2}\, dz\,, \qquad h)\ \int_{|z-1|=2} \frac{dz}{z^2-2i}\,, \qquad i)\ \int_{|z|=2} \frac{dz}{z^2(z^2+16)}\,,$$

$$j)\ \int_{|z|=3/2} \frac{\sinh 5z}{(1+z^2)z^2}\, dz\,, \qquad k)\ \int_{|z-z_0|=r} \frac{2z - \sin 2z + 2(z-z_0)\cos^2 z}{(z-z_0)^2}\, dz\,,$$

$$l)\ \int_{|z-2|=1} \frac{e^z}{z}\, dz\,, \qquad m)\ \int_{|z|=1} \frac{e^z}{z}\, dz\,, \qquad n)\ \int_{|z|=1} \frac{\sin z}{z}\, dz\,,$$

$$o)\ \int_{|z|=1} \frac{e^{3z}}{(z-1/2)^5}\, dz\,, \qquad p)\ \int_{|z|=2} \frac{z^2}{z^3-1}\, dz\,, \qquad q)\ \int_{|z|=2} \frac{z^2}{(z-1)^3}\, dz\,,$$

$$r)\ \int_{|z|=3} \frac{e^{zt}}{z^2+1}\, dz\,, \qquad s)\ \int_{|z|=3} \frac{z\, e^{zt}}{(z+1)^3}\, dz\,.$$

Solutions: *a)* $2\pi i$, *b)* $2\pi i$, *c)* $2\pi i(2+e^2)$, *d)* $2\pi i$, *e)* 0, *f)* $\pi i/2$, *g)* $4\pi i$, *h)* $\frac{\sqrt{2}}{2}\,\pi e^{i\pi/4}$, *i)* 0, *j)* $2\pi i(5-\sin 5)$, *k)* $4\pi i(1+\sin^2 z_0)$, *l)* 0, *m)* $2\pi i$, *n)* 0, *o)* $\frac{27}{4}\,\pi i e^{3/2}$, *p)* $2\pi i$, *q)* $2\pi i$, *r)* $2\pi i\, \sin t$, *s)* $\pi i(2t-t^2)\, e^{-t}$.

6.2. Compute the following integral in terms of $r \in (0,1) \cup (1,2) \cup (2,\infty)$:

$$I = \int_{|z|=r} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz\,.$$

Solution: $I = 0$, if $0 < r < 1$; $I = 2\pi i$, if $1 < r < 2$; $I = 4\pi i$, if $r > 2$.

6.3. Compute the following integral along the curve γ in the following cases:

$$\int_\gamma \frac{e^z}{z(1-z)^3}\, dz\,,$$

a) γ is any curve with $n(\gamma,0)=1$, $n(\gamma,1)=0$.
b) γ is any curve with $n(\gamma,0)=0$, $n(\gamma,1)=1$.
c) γ is any curve with $n(\gamma,0)=1$, $n(\gamma,1)=1$.
d) γ is any curve with $n(\gamma,0)=0$, $n(\gamma,1)=0$.

Solutions: a) $2\pi i$, b) $-e\pi i$, c) $(2-e)\pi i$, d) 0.

6.4. a) Prove the *Gauss' mean value theorem*: If f is holomorphic on a domain containing $\{|z-z_0| \le r\}$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt\,.$$

Hint: Use Cauchy's integral formula along the circumference with center z_0 and radius r.

b) Prove that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta \;=\; 2\pi\,.$$

Hint: Apply the previous item to the function $f(z) = \cos z$.

c) Deduce the mean value theorem for harmonic functions: If u is a harmonic function on a domain containing $\{|z - z_0| \le r\}$, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})\, dt\,.$$

Hint: Every harmonic function on a simply connected set D has a conjugate harmonic function on D.

6.5. a) Let $f(z)$ be an holomorphic function on $\{z \in \mathbf{C} : |z| < R_0\}$. Prove that if $|a| < R < R_0$, and $\gamma = \{z \in \mathbf{C} : |z| = R\}$, then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{R^2 - |a|^2}{(z-a)(R^2 - z\overline{a})}\, f(z)\, dz\,.$$

Hint: Write the rational function on z as a sum of simple fractions and apply Cauchy's integral formula.

b) Deduce the *Poisson formula* for holomorphic functions (by using a)), for $0 \le r < R$ and $0 \le \theta < 2\pi$:

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, f(Re^{i\varphi})\, d\varphi\,.$$

Hint: Consider $a = re^{i\theta}$ and $z = Re^{i\varphi}$.

c) Deduce the *Poisson formula* for harmonic functions u on $\{z \in \mathbf{C} : |z| < R_0\}$ (by using b)) for $0 \le r < R$ and $0 \le \theta < 2\pi$:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\, u(Re^{i\varphi})\, d\varphi\,.$$

Hint: Every harmonic function on a simply connected set D has a conjugate harmonic function on D.

6.6. a) Assume that $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ are simple closed curves such that $\gamma_1, \gamma_2, \ldots, \gamma_n$ are contained in the interior of γ. If f is an holomorphic function on the closure of the region Ω with $\partial\Omega = \cup_{k=1}^n \gamma_k \cup \gamma$, prove that:

$$\int_\gamma f(z)\; dz \;=\; \sum_{k=1}^n \int_{\gamma_k} f(z)\; dz\;.$$

Hint: Add some paths to the integral in order to write Ω as a union of simply connected domains.

b) Under the hypotheses in the previous item, prove the following Cauchy's formula, for every $a \in \Omega$:

$$\int_\gamma \frac{f(z)\; dz}{z-a} \;-\; \sum_{k=1}^n \int_{\gamma_k} \frac{f(z)\; dz}{z-a} \;=\; 2\pi i\, f(a)\,.$$

Hint: Use the previous item with the function $(f(z) - f(a))/(z-a)$.

c) Let f be an holomorphic function on $\{z \in \mathbf{C} : 0 < |z| < R\}$. Prove that the value of $\int_0^{2\pi} f(re^{it})dt$, with $0 < r < R$, is independent of r. If f is holomorphic on the whole disk, compute the value of $\int_0^{2\pi} f(re^{it})dt$.

Hint: Use item a).

6.7. Prove *Liouville's theorem*: If f is an entire function (i.e., holomorphic on $\mathbf{C}$) satisfying $|f(z)| \le M$, for every $z \in \mathbf{C}$, then f is constant.

Hint: Prove:

a) $|f'(a)| \le M/r^2$ for every $a \in \mathbf{C}$.

Hint: Use the Cauchy's inequality for f'.

b) $f'(a) = 0$ for every $a \in \mathbf{C}$.

Hint: Take the limit in the inequality of item a) as r goes to some appropriate value.

6.8. Prove the *Fundamental Theorem of Algebra*: Every polynomial (with complex coefficients) of degree $n \geq 1$ has n complex zeros (taking into account the multiplicity of its zeros).

Hint: It suffices to prove that the polynomial P has a zero. Seeking for a contradiction assume that $P \neq 0$ and apply Liouville's theorem to the function $1/P$.

6.9. Let f be an entire function. Prove the following statements by using Liouville's theorem:
a) If $|f| \geq 1$, then f is constant.
b) If $\operatorname{Re} f \geq 0$, then f is constant.
c) If $\operatorname{Im} f \leq 1$, then f is constant.
d) If $\operatorname{Re} f$ does not have zeros, then f is constant.
e) If there exists an straight line that does not intersect the image of f, then f is constant.

Hint: Find a bounded entire function in terms of f.

6.10. Prove that the following functions are entire if they are defined in an appropriate way at their singular points:

a) $\dfrac{\sin z}{z}$, b) $\dfrac{e^z - 1 - z}{z^2}$, c) $\dfrac{\sin(\pi z)}{z^3 - z}$, d) $\dfrac{\sin(\pi z^2)}{\sin(\pi z)}$.

Hint: Prove that $\lim_{z \to a} (z - a) f(z) = 0$ at each singular point a.

6.11. Prove that $F(z) = \int_0^1 e^{-z^2 x^2}\, dx$ is an entire function, and compute $F'(z)$.

Hint: For the first statement you can use Morera's theorem. For the second one, prove that you can compute the derivative inside the integral.

6.12. If $f : [0, \infty) \longrightarrow \mathbf{C}$ is a function with $\lim_{x \to \infty} f(x)\, e^{-ax} = 0$ for some $a \in \mathbf{R}$ and $f \in L^1([0, n])$ for every n, then the Laplace transform of f is defined as

$$Lf(z) = \int_0^\infty f(x)\, e^{-zx}\, dx.$$

Prove that $f(x)\, e^{-zx}$ is integrable on $[0, \infty)$ for each complex number z with $\operatorname{Re} z > a$ and that Lf is holomorphic on the halfplane $\{z \in \mathbf{C} : \operatorname{Re} z > a\}$. Prove that the properties of the Laplace transform as a function of a real variable also hold for the Laplace transform as a function of a complex variable.

Hint: For the first statement you can use Morera's theorem.

6.13. It can be proved that if f is a "good enough" function, then we can obtain f from its Laplace transform by the following Mellin's inverse formula:

$$f(x) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\gamma_{t,T}} e^{zx}\, (Lf)(z)\, dz\,,$$

where if $\lim_{s \to \infty} f(s)\, e^{-as} = 0$, then $\gamma_{t,T}$ is the vertical segment $\gamma_{t,T} = \{z \in \mathbf{C} : \operatorname{Re} z = t, \operatorname{Im} z \in [-T, T]\}$ oriented starting at $t - iT$ and ending at $t + iT$, with $t > a$. Alternatively, we can choose t greater than the real part of all singularities of $F(z)$.

Compute $f(x)$ if:

$$(Lf)(z) = \frac{1}{z - 3}\,, \quad (Lf)(z) = \frac{z}{(z-1)^2 (z^2 + 3z - 10)}\,, \quad (Lf)(z) = \frac{1}{z^2 (z^2 + 2z + 2)}\,,$$

and check that its value is independent on the choice of $t > a$.

Hint: Apply Cauchy's integral formula for an appropriate closed curve containing $\gamma_{t,T}$ and a part of circumference C_T joining the endpoints of $\gamma_{t,T}$. Prove that the integral along C_T goes to 0 as T goes to ∞.

6.14. Let f be an holomorphic function on a simply connected domain D, such that $f(z) \neq 0$ for every $z \in D$.

a) Prove that for any $z_0 \in D$, the function $L(z) = \int_{z_0}^{z} f'(w)/f(w)\, dw$ is well defined (i.e., the value of $L(z)$ is independent of the curve joining z_0 with z) and so, it is possible to define the function $\log f(z)$ in such a way that it is holomorphic on D.

b) Prove that for any $\alpha \in \mathbf{C}$, it is possible to define the function $f(z)^\alpha$ in such a way that it is holomorphic on D.

c) Do items a) or b) hold in general if D is not simply connected?

d) Find a domain D such that $(1 - z^2)^{-1/2}$ can be defined as a holomorphic function on D. Is $\arcsin z$ holomorphic on that domain?

e) Find a domain D such that $\arctan z$ can be defined as a holomorphic function on D.

Hints: a) Use that $\int_\gamma^z f'(w)/f(w)\, dw = 0$ for any closed curve γ in D. b) Use the previous item.

6.15. Let f be a meromorphic function on a domain D (i.e., a holomorphic function on D except for a set of isolated points, which are poles of f).

a) Prove that if f has a zero of order k at a, then the function

$$\frac{f'(z)}{f(z)} - \frac{k}{z-a}$$

is holomorphic on a neighborhood of a.

b) Prove that if f has a pole of order k at a, the function

$$\frac{f'(z)}{f(z)} + \frac{k}{z-a}$$

is holomorphic on a neighborhood of a.

Hints: a) If f has a zero of order k at a, then $f(z) = (z-a)^k g(z)$ where g is a holomorphic function on a neighborhood of a with $g(a) \neq 0$. b) If f has a pole of order k at a, then $f(z) = g(z)/(z-a)^k$ where g is a holomorphic function on a neighborhood of a with $g(a) \neq 0$.

6.16. a) Let f be a meromorphic function on a simply connected domain D with zeros $a_1, a_2, \ldots, a_r$ and poles $b_1, b_2, \ldots, b_s$ (in each list appear the zeros and the poles taking into account their multiplicities, i.e., if a zero or a pole has order k, it appears k times in the list). Prove that if γ is a closed curve contained on D with $\gamma \cap \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s\} = \emptyset$, and $\Gamma = f \circ \gamma$ is the image of γ by f, then

$$\frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{r} n(\gamma, a_j) - \sum_{k=1}^{s} n(\gamma, b_k)\,.$$

This result is known as the *argument principle*. In particular, if γ is a Jordan curve in D enclosing a simply connected domain $D_\gamma \subset D$, the argument principle gives that the above integral is equal to the number of zeros of f on D_γ minus the number of poles of f on D_γ (taking into account their multiplicities).

Hint: In order to prove the first equality, note that if $\gamma(t)$ is a parametrization of γ, then $f(\gamma(t))$ is a parametrization of Γ. In order to prove the second equality, use the previous exercise to show that the following function is holomorphic on the domain D

$$\frac{f'(z)}{f(z)} - \sum_{j=1}^{r} \frac{1}{z-a_j} + \sum_{k=1}^{s} \frac{1}{z-b_k}\,.$$

b) Use the previous item to compute

$$i)\ \int_{|z|=2} \tan z\, dz\,, \qquad ii)\ \int_{|z|=2} \frac{dz}{\sin z\, \cos z}\,.$$

Hints: *i)* Consider the function $f(z) = \cos z$. *ii)* Consider the function $f(z) = \operatorname{cosec} 2z + \operatorname{cotan} 2z$.

Solutions: *i)* $2\pi i$, *ii)* $-2\pi i$.

6.17. Let f, g be two holomorphic functions on a domain D, and γ a Jordan curve (i.e., a simple closed curve) in D surrounding a simply connected domain $D_\gamma \subset D$. If f, g satisfy the inequality $|f(z) - g(z)| < |f(z)|$ for every $z \in \gamma$, prove that:

a) The function $F = g/f$ does not have zeros nor poles in the curve γ, i.e., f and g do not have zeros in γ.

b) If Γ is the image by F of γ, then $\int_\Gamma dw/w = 0$.

Hint: Since $|g(z)/f(z) - 1| < 1$ on γ, we have $|w - 1| < 1$ on Γ.

c) Prove that $f(z)$ and $g(z)$ have the same number of zeros on D_γ (taking into account their multiplicities). This result is known as *Rouché's theorem.*

Hint: Use the argument principle.

6.18. Apply Rouché's theorem in order to solve the following problems:

a) How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the unit disk $\mathbf{D} = \{|z| < 1\}$?

Hint: Consider $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$ and choose $f(z)$ as the monomial of $g(z)$ with greatest modulus on $\{|z| = 1\}$.

b) How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $\{|z| < 2\}$?

c) How many roots does the equations $z^9 - 2z^6 + z^2 - 8z - 2 = 0$, $2z^5 - z^3 + 3z^2 - z + 8 = 0$, $z^7 - 5z^4 + z^2 - 2 = 0$, have in $\mathbf{D}$?

d) How many roots does the equation $z^4 - 6z + 3 = 0$ have in the disk $\{|z| < 2\}$? And in $\mathbf{D}$? And in the annulus $\{1 < |z| < 2\}$?

e) How many roots does the equation $z^4 - 5z + 1 = 0$ have in $\mathbf{D}$? And in the annulus $\{1 < |z| < 2\}$?

f) How many roots does the equation $z^4 - 8z + 10 = 0$ have in $\mathbf{D}$? And in the annulus $\{1 < |z| < 3\}$?

g) How many roots does the equation $z^n + az^2 + bz + c = 0$ have in $\mathbf{D}$, if $|a| > |b| + |c| + 1$ and $n \in \mathbf{N}$?

h) How many roots does the equation $z = f(z)$ have in $\mathbf{D}$, if f is an holomorphic function satisfying $|f(z)| < 1$ if $|z| \leq 1$?

i) How many roots does the equation $e^z - 4z^n + 1 = 0$ have in $\mathbf{D}$, if $n \in \mathbf{N}$?

Solutions: a) 3, b) 7, c) 1, 0, 4, d) 4, 1, 3, e) 1, 3, f) 0, 4, g) 2, h) 1, i) n.

6.19. Let f be an holomorphic function on $\mathbf{D}$ satisfying $|f(z)| < 1$ for every $z \in \mathbf{D}$ and $f(0) = 0$. Prove that:

a) The function $g(z) = f(z)/z$ is holomorphic on $\mathbf{D}$.

b) For each $0 < r < 1$ we have $|g(z)| \leq 1/r$ if $|z| \leq r$.

Hint: Use the maximum modulus principle.

c) $|f(z)| \leq |z|$ for every $z \in \mathbf{D}$ and $|f'(0)| \leq 1$.

Hint: Use the previous item.

d) If there exists a point $z_0 \in \mathbf{D}$ such that $|f(z_0)| = |z_0|$ (or $|f'(0)| = 1$), then $f(z) = cz$ where c is a complex number with $|c| = 1$.

Hint: Use the maximum modulus principle.

These results are known as *Schwarz Lemma.*

6.20. Prove the *minimum modulus principle*: If $f(z)$ is an holomorphic function on the domain D, $f(z) \neq 0$ for every $z \in D$, and $|f(z)|$ attains its minimum value at a point in D, then $f(z)$ is constant.

Hint: Use the maximum modulus principle for an appropriate holomorphic function.

6.21. Study if there exists an holomorphic function on $\mathbf{D}$ such that on the points $1/n$ $(n = 1, 2, 3, \dots)$ take the values:

a) $0, 1, 0, 1, 0, 1, \dots$

b) $0, 1/2, 0, 1/4, 0, 1/6, \dots, 0, 1/(2k), \dots$

c) $1/2, 1/2, 1/4, 1/4, 1/6, 1/6, \dots, 1/(2k), 1/(2k), \dots$

d) $1/2, 2/3, 3/4, 4/5, 5/6, 6/7, \dots, n/(n+1), \dots$

Hint: Recall that $f = 0$ is the unique holomorphic function on $\mathbf{D}$ such that $f(a_n) = 0$ for a sequence with $\lim_{n\to\infty} a_n = 0$.