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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.7: Laurent series

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1.7. LAURENT SERIES

7.1. Compute the power series about 0 of the following functions and find the radius of convergence in each case:

$$a)\ \sin^2 z\,, \qquad b)\ \cosh^2 z\,, \qquad c)\ \sqrt{z+i}\,, \qquad d)\ \frac{6z}{z^2-4z+13}\,, \qquad e)\ \frac{z^2}{(z+1)^2}\,,$$

$$f)\ \log\frac{1+z}{1-z}\,, \qquad g)\ \arctan z\,, \qquad h)\ \arcsin z\,, \qquad i)\ \operatorname{arsinh} z\,, \qquad j)\ \sin z^2\,,$$

$$k)\ \frac{\sin z}{z}\,, \qquad l)\ \frac{e^z-1-z}{z^2}\,, \qquad m)\ \frac{1-\cos z^3}{z^5}\,, \qquad n)\ (1-z)^{-p} \quad (p\in\mathbf{C})\,, \qquad o)\ \int_0^1 e^{t^2z^2}\,dt\,,$$

Solutions: *a)* $\sum_{n=1}^\infty (-1)^{n+1}2^{2n-1}z^{2n}/(2n)!$, $R=\infty$; *b)* $1+\sum_{n=1}^\infty 2^{2n-1}z^{2n}/(2n)!$, $R=\infty$;
c) $e^{i\pi/4}\sum_{n=0}^\infty (-i)^n\binom{1/2}{n}z^n$, $R=1$, if we choose $\sqrt{i}=e^{i\pi/4}$;
d) $i\sum_{n=1}^\infty ((2-3i)^n-(2+3i)^n)z^n/13^n$, $R=\sqrt{13}$; *e)* $\sum_{n=2}^\infty (-1)^n(n-1)z^n$, $R=1$;
f) $2\sum_{n=0}^\infty z^{2n+1}/(2n+1)$, $R=1$; *g)* $\sum_{n=0}^\infty (-1)^n z^{2n+1}/(2n+1)$, $R=1$;
h) $\sum_{n=0}^\infty (-1)^n\binom{-1/2}{n}z^{2n+1}/(2n+1)$, $R=1$; *i)* $\sum_{n=0}^\infty \binom{-1/2}{n}z^{2n+1}/(2n+1)$, $R=1$;
j) $\sum_{n=0}^\infty (-1)^n z^{4n+2}/(2n+1)!$, $R=\infty$; *k)* $\sum_{n=0}^\infty (-1)^n z^{2n}/(2n+1)!$, $R=\infty$;
l) $\sum_{n=0}^\infty z^n/(n+2)!$, $R=\infty$; *m)* $\sum_{n=1}^\infty (-1)^{n+1}z^{6n-5}/(2n)!$, $R=\infty$;
n) $\sum_{n=0}^\infty (-1)^n\binom{-p}{n}z^n=\sum_{n=0}^\infty\binom{p+n-1}{n}z^n$, $R=1$; *o)* $\sum_{n=0}^\infty z^{2n}/((2n+1)n!)$, $R=\infty$.

7.2. Compute the power series about 1 of the following functions and find the radius of convergence in each case:

$$a)\ \frac{z^2}{(z+1)^2}\,, \qquad b)\ \frac{4z}{z^2-2z+5}\,, \qquad c)\ z^{1/3}\,, \qquad d)\ \sin(2z-z^2)\,.$$

Solutions: *a)* $1/4+\sum_{n=1}^\infty (-1)^n(n-3)(z-1)^n/2^{n+2}$, $R=2$; *b)* $\sum_{n=0}^\infty (-1)^n((z-1)^{2n}+(z-1)^{2n+1})/4^n$, $R=2$; *c)* $\sum_{n=0}^\infty\binom{1/3}{n}(z-1)^n$, $R=1$; *d)* $\sum_{n=0}^\infty \sin(1+n\pi/2)(z-1)^{2n}/n!$, $R=\infty$.

7.3. Compute the terms with degree less than three in the power series about 0 of the following functions (without compute their derivatives):

$$a)\ (\log(1-z))^2\,, \qquad b)\ e^{z/(1-z)}\,, \qquad c)\ e^{e^z}\,, \qquad d)\ (1+z)^z\,,$$

$$e)\ e^{z\,\sin z}\,, \qquad f)\ \sqrt{\cos z}\,, \qquad g)\ \frac{z}{e^z-1}\,.$$

Solutions: *a)* z^2, *b)* $1+z+3z^2/2$, *c)* $e+ez+ez^2$, *d)* $1+z^2$, *e)* $1+z^2$, *f)* $1-z^2/4$, *g)* $1-z/2+z^2/12$.

7.4. a) Let $f(z)=\sum_{n=0}^\infty a_n\,(z-a)^n$ for $|z-a|<R$. Prove the *Parseval's formula*:

$$\int_0^{2\pi}|f(a+re^{it})|^2\,dt=2\pi\sum_{n=0}^\infty |a_n|^2\,r^{2n}\,, \qquad \text{if } 0\le r<R\,.$$

Hint: $|f(a+re^{it})|^2=(\sum_{n=0}^\infty a_nr^ne^{itn})(\sum_{m=0}^\infty \overline{a_m}r^me^{-itm})$.

b) If $M(r)=\max\{|f(z)|:|z-a|=r\}$, prove

$$\sum_{n=0}^\infty |a_n|^2\,r^{2n}\le M(r)^2\,, \qquad \text{if } 0\le r<R\,.$$

Hint: Use the previous item.

c) Prove that we have for every $n \in \mathbf{N}$

$$|a_n| \le \frac{M(r)}{r^n}\,, \qquad \text{if } 0 < r < R\,.$$

Hint: Use item b).

d) Assume that there exist $n \in \mathbf{N}$ and $0 < r < R$ such that $|a_n| = M(r)/r^n$. Prove that $f(z) = a_n(z-a)^n$.

Hint: Use item b).

7.5. Compute the Laurent series about $z_0 = 0$ and $z_0 = \infty$ (if it is possible) of the following functions, and find the annuli in which they are convergent:

$$a)\ \frac{1}{z^2-3z+2}\,, \qquad b)\ \frac{1}{z} + \frac{1}{(z-1)^2} + \frac{1}{z+2}\,,$$

$$c)\ \sin\frac{1}{z}\,, \qquad d)\ \frac{1}{z(1-z)}\,, \qquad e)\ \frac{1}{(z-a)^k}\ \ (a \neq 0, k \in \mathbf{N})\,,$$

$$f)\ z^2 e^{1/z}\,, \qquad g)\ \log\frac{z-a}{z-b}\,, \qquad h)\ \frac{\arctan z}{z^4}\,,$$

$$i)\ \frac{1}{(z-a)(z-b)}\,, \quad \text{in terms of the values of } a, b \in \mathbf{C}.$$

Solutions:

a) $1/(z^2-3z+2) = \sum_{n=0}^{\infty}(2^{n+1}-1)z^n/2^{n+1}$, if $|z| < 1$; $1/(z^2-3z+2) = \sum_{n=0}^{\infty}(2^n-1)/z^{n+1}$, if $|z| > 2$.

b) $1/z + 1/(z-1)^2 + 1/(z+2) = 1/z + \sum_{n=0}^{\infty}(n+1+(-1)^n/2^{n+1})\,z^n$, if $0 < |z| < 1$;
$1/z + 1/(z-1)^2 + 1/(z+2) = 2/z + \sum_{n=1}^{\infty}(n+(-1)^n 2^n)/z^{n+1}$, if $|z| > 2$.

c) $\sin(1/z) = \sum_{n=0}^{\infty}(-1)^n/((2n+1)!\,z^{2n+1})$, if $0 < |z| < \infty$.

d) $1/(z(1-z)) = \sum_{n=-1}^{\infty} z^n$, if $0 < |z| < 1$; $1/(z(1-z)) = -\sum_{n=2}^{\infty} 1/z^n$, if $|z| > 1$.

e) $1/(z-a)^k = (-1/a)^k \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}(z/a)^n$, if $|z| < |a|$;
$1/(z-a)^k = (1/z)^k \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}(a/z)^n$, if $|z| > |a|$.

f) $z^2 e^{1/z} = \sum_{n=0}^{\infty} 1/(n!\,z^{n-2})$, if $0 < |z| < \infty$.

g) $\log(z-a)/(z-b) = \log(a/b) + \sum_{n=1}^{\infty}(1/b^n - 1/a^n)\,z^n/n$, if $|z| < \min\{|a|,|b|\}$ $(a, b \neq 0)$;
$\log(z-a)/(z-b) = \sum_{n=1}^{\infty}(b^n-a^n)/(nz^n)$, if $|z| > \max\{|a|,|b|\}$.

h) $z^{-4}\arctan z = \sum_{n=0}^{\infty}(-1)^n z^{2n-3}/(2n+1)$, if $0 < |z| < 1$;
$z^{-4}\arctan z = \pi z^{-4}/2 + \sum_{n=0}^{\infty}(-1)^{n+1}z^{-2n-5}/(2n+1)$, if $|z| > 1$.

$$i)\qquad \frac{1}{(z-a)(z-b)} = \begin{cases} \dfrac{1}{z^2}\,, & \text{if } a=b=0,\ 0<|z|<\infty,\\[2mm] \displaystyle\sum_{n=0}^{\infty}\frac{(n+1)z^n}{a^{n+2}}\,, & \text{if } a=b\neq 0,\ |z|<|a|,\\[2mm] \displaystyle -\sum_{n=-1}^{\infty}\frac{z^n}{a^{n+2}}\,, & \text{if } 0=b\neq a,\ |z|<|a|,\\[2mm] \displaystyle \frac{1}{a-b}\sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}-a^{n+1}}\,, & \text{if } 0\neq a\neq b\neq 0,\ |z|<\min\{|a|,|b|\}, \end{cases}$$

$$\frac{1}{(z-a)(z-b)} = \begin{cases} \dfrac{1}{z^2}\,, & \text{if } a=b=0,\ 0<|z|<\infty,\\[2mm] \displaystyle\sum_{n=2}^{\infty}\frac{(n-1)a^{n-2}}{z^n}\,, & \text{if } a=b\neq 0,\ |a|<|z|<\infty,\\[2mm] \displaystyle\sum_{n=2}^{\infty}\frac{a^{n-2}}{z^n}\,, & \text{if } 0=b\neq a,\ |a|<|z|<\infty,\\[2mm] \displaystyle \frac{1}{a-b}\sum_{n=2}^{\infty}\frac{a^{n-1}-b^{n-1}}{z^n}\,, & \text{if } 0\neq a\neq b\neq 0,\ \max\{|a|,|b|\}<|z|<\infty. \end{cases}$$

7.6. Compute the Laurent series about the indicated points of the following functions, and find the annuli in which they are convergent:

$$a)\ \frac{1}{(z^2+1)^2}\,,\qquad z_0=i,\qquad b)\ e^{1/(1-z)}\,,\qquad z_0=1,$$
$$c)\ \frac{\sin z}{(z-\pi)^2}\,,\qquad z_0=\pi,\qquad d)\ z^2\,\sin\frac{1}{z-1}\,,\qquad z_0=1,$$
$$e)\ \sin\frac{z}{1-z}\,,\qquad z_0=1,\qquad f)\ \cos\frac{z^2-4z}{(z-2)^2}\,,\qquad z_0=2.$$

Solutions: $a)\ \dfrac{1}{(z^2+1)^2}=\displaystyle\sum_{n=-2}^{\infty}\frac{(n+3)\,i^n}{2^{n+4}}\,(z-i)^n$, if $0<|z-i|<2$.

$b)\ e^{1/(1-z)}=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\frac{1}{(z-1)^n}$, if $0<|z-1|<\infty$.

$c)\ \dfrac{\sin z}{(z-\pi)^2}=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\,(z-\pi)^{2n-1}$, if $0<|z-\pi|<\infty$.

$d)\ z^2\sin\dfrac{1}{z-1}=z-1+\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(4n^2+10n+5)}{(2n+3)!(z-1)^{2n+1}}+\sum_{n=0}^{\infty}\frac{2\,(-1)^n}{(2n+1)!(z-1)^{2n}}$, if $0<|z-1|<\infty$.

$e)\ \sin\dfrac{z}{1-z}=\cos 1\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!(z-1)^{2n+1}}+\sin 1\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!(z-1)^{2n}}$, if $0<|z-1|<\infty$.

$f)\ \cos\dfrac{z^2-4z}{(z-2)^2}=\cos 1\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n4^{2n}}{(2n)!(z-2)^{4n}}-\sin 1\sum_{n=0}^{\infty}\frac{(-1)^n4^{2n+1}}{(2n+1)!(z-2)^{4n+2}}$, if $0<|z-2|<\infty$.

7.7. Compute the Laurent series on the indicated regions Ω of the following functions:

$$a)\ \frac{1}{(z-a)(z-b)}\,,\quad (0<|a|<|b|)\,,\ \Omega=\{z:|a|<|z|<|b|\}\,,$$
$$b)\ \frac{z^2-2z+5}{(z-2)(z^2+1)}\,,\quad \Omega=\{z:1<|z|<2\}\,,$$
$$c)\ \frac{1}{(z-2)^2}\,,\quad \Omega_1=\{z:|z|<2\}\,,\ \Omega_2=\{z:|z|>2\}\,,$$
$$d)\ e^{z+1/z}\,,\quad \Omega=\{z:0<|z|<\infty\}\,,$$
$$e)\ \sin z\,\sin\frac{1}{z}\,,\quad \Omega=\{z:0<|z|<\infty\}\,.$$

Solutions: $a)$ $1/((z-a)(z-b))=(a-b)^{-1}\sum_{n=0}^{\infty}(z^n/b^{n+1}+a^n/z^{n+1})$.

$b)$ $(z^2-2z+5)/((z-2)(z^2+1))=\sum_{n=0}^{\infty}(2(-1)^{n+1}/z^{2n+2}-z^n/2^{n+1})$.

$c)$ $1/(z-2)^2=(-1/2)^2\sum_{n=0}^{\infty}(n+1)(z/2)^n$, if $|z|<2$;
$1/(z-2)^2=(1/z)^2\sum_{n=0}^{\infty}(n+1)(2/z)^n$, if $|z|>2$.

$d)$ $e^{z+1/z}=\sum_{n=-\infty}^{\infty}c_nz^n$ with $c_n=\sum_{k=k_n}^{\infty}1/(k!(n+k)!)$, $k_n=\max\{0,n\}$.

$e)$ $\sin z\,\sin(1/z)=\sum_{n=-\infty}^{\infty}(-1)^nc_nz^{2n}$ with $c_n=\sum_{k=k_n}^{\infty}1/((2k+1)!(2k-2n+1)!)$, $k_n=\max\{0,n\}$.

7.8. L'Hôpital-Bernoulli's rule for complex functions:

a) Let f and g be two analytic functions with a zero of order k at z_0. Prove that f/g has a removable singularity at z_0 and

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}\,.$$

b) Let $f(z)$ and $g(z)$ be analytic functions at z_0 such that $f(z_0)\neq 0$, $g(z_0)=0$ and $g'(z_0)\neq 0$. Prove that $h=f/g$ has a simple pole at z_0 and that $\mathrm{Res}\,(h,z_0)=f(z_0)/g'(z_0)$.

c) Let $f(z)$ and $g(z)$ be analytic functions at z_0 which have at z_0 a zero of order k and $k+1$, respectively. Prove that $h = f/g$ has a simple pole at z_0 and $\mathrm{Res}\,(h, z_0) = (k+1)\, f^{(k)}(z_0)/g^{(k+1)}(z_0)$.

d) Let f and g be analytic functions at z_0 which have a zero at z_0 of order n and k, respectively, with $k \geq n$. Prove that $h = f/g$ has a pole at z_0 of order $k-n$. Find a formula (as in item b)) for the residue of h on z_0.

e) Think what happens on each one of the previous cases if f and/or g have at z_0 a pole instead of a zero.

Hint: Write the Taylor (or Laurent) series of f and g.

7.9. Determine the nature of the singularities of each of the following functions (including the point ∞, that is, the singularity of $f(1/z)$ at $z = 0$) and compute the residue at the (finite) singularities:

$$a)\ \frac{e^z}{1+z^2}\,, \qquad b)\ z\,e^{-z}\,, \qquad c)\ \frac{e^z}{z(1-e^{-z})}\,, \qquad d)\ \frac{z-\sin z}{z^3}\,,$$

$$e)\ (z-3)\,\sin\frac{1}{z+2}\,, \qquad f)\ \frac{1-\cos z}{z}\,, \qquad g)\ \frac{z^2}{e^{z^4}}\,, \qquad h)\ \frac{1}{z}\,\cosh\frac{1}{z}\,,$$

$$i)\ \cos\left(z^2+\frac{1}{z^2}\right)\,, \qquad j)\ e^{z/(z-2)}\,, \qquad k)\ e^{-z}\,\cos\frac{1}{z}\,, \qquad l)\ \frac{1}{z^2}+\sin\frac{1}{z}\,,$$

$$m)\ \frac{e^{2z}}{(z-1)^3}\,, \qquad n)\ \frac{z^{2n}}{(1+z)^n}\,, \qquad o)\ \frac{\mathrm{cotan}\, z\, \mathrm{cotanh}\, z}{z^3}\,, \qquad p)\ \mathrm{cosec}\,\frac{1}{z}\,,$$

$$q)\ \frac{z^4}{1+z^4}\,, \qquad r)\ \frac{1}{z^3-z^5}\,, \qquad s)\ \frac{1-e^z}{1+e^z}\,,$$

$$t)\ \frac{1}{\sin z - \sin a}\,, \qquad u)\ z\,e^{z/(1-z)} - \frac{1}{(z-1)^2}\,\sin\frac{1}{z}\,, \qquad v)\ \frac{\tan z}{z^n}\,.$$

Solutions: *a*) At $z = i, -i$, it has simple poles with residues $-ie^{i}/2$, $ie^{-i}/2$; at $z = \infty$ it has an essential singularity. *b*) At $z = \infty$ it has an essential singularity. *c*) At $z = 2k\pi i$ with $k \in \mathbf{Z}\backslash\{0\}$, it has a simple pole with residue $1/(2k\pi i)$; at $z = 0$ it has a double pole with residue $3/2$; at $z = \infty$ it has a non-isolated singularity. *d*) $z = 0$ is a removable singularity; $z = \infty$ is an essential singularity. *e*) $z = -2$ is an essential singularity with residue -5; $z = \infty$ is a removable singularity. *f*) $z = 0$ is a removable singularity; $z = \infty$ is an essential singularity. *g*) $z = \infty$ is an essential singularity. *h*) $z = 0$ is an essential singularity with residue 1; $z = \infty$ is a removable singularity. *i*) $z = 0$, $z = \infty$ are essential singularities; the residue at $z = 0$ is 0. *j*) $z = 2$ is an essential singularity with residue $2e$; $z = \infty$ is a removable singularity. *k*) $z = 0$, $z = \infty$ are essential singularities; the residue at $z = 0$ is $\sum_{n=1}^{\infty}(-1)^{n+1}/((2n-1)!(2n)!)$. *l*) $z = 0$ is an essential singularity with residue 1; $z = \infty$ is a removable singularity. *m*) $z = 1$ is a pole of order 3 with residue $2e^2$; $z = \infty$ is an essential singularity. *n*) $z = -1$ is a pole of order n with residue $(-1)^n\binom{2n}{n+1}$; $z = \infty$ is a pole of order n. *o*) $z = 0$ is a pole of order 5 with residue $4B_4/3 - 4B_2^2 = -7/45$ (B_4 and B_2 are the fourth and second Bernoulli numbers, see Exercise 7.10); the points $z = k\pi$, $z = k\pi i$ ($k \in \mathbf{Z}\backslash\{0\}$) are simple poles with residue $(k\pi)^{-3}(e^{2k\pi}+1)/(e^{2k\pi}-1)$; $z = \infty$ is a non-isolated singularity. *p*) $z = 0$ is a non-isolated singularity; $z = \infty$ is an essential singularity; the points $z = (k\pi)^{-1}$ ($k \in \mathbf{Z}\backslash\{0\}$) are simple poles with residue $(-1)^{k+1}(k\pi)^{-2}$. *q*) the points $z_1 = (1+i)/\sqrt{2}$, $z_2 = (-1+i)/\sqrt{2}$, $z_3 = (-1-i)/\sqrt{2}$, $z_4 = (1-i)/\sqrt{2}$ (the fourth roots of -1) are simple poles with residue $\mathrm{Res}\,(f, z_j) = -1/\prod_{i\neq j}(z_j - z_i)$; $z = \infty$ is a removable singularity. *r*) $z = 0$ is a pole of order 3 with residue 1; $z = 1$, $z = -1$ are simple poles with residue $-1/2$; $z = \infty$ is a removable singularity. *s*) the points $z = (2k+1)\pi i$ are simple poles with residue -2; $z = \infty$ is a non-isolated singularity. *t*) $z = \infty$ is a non-isolated singularity; if a is not an integer multiple of π (if $\sin a \neq 0$) the points $z = a + 2k\pi$ are simple poles with residue $\sec a$; if a is an integer multiple of π (if $\sin a = 0$) the points $z = k\pi$ are simple poles with residue $(-1)^k$. *u*) $z = 0$ is an essential singularity with residue $-\cos 1$; $z = 1$ is an essential singularity with residue $-\cos 1 - 1/(2e)$; $z = \infty$ is a simple pole. *v*) $z = 0$ is a removable singularity if $n = 1$; if $n > 1$, $z = 0$ is a pole of order $n-1$ with residue: 0, if n is odd and $(-1)^{n/2}2^n(1-2^n)B_n/(n(n-1))$ (B_n is the n-th Bernoulli number, see Exercise 7.10); the points $z = (2k+1)\pi/2$ are simple poles with residue $-(2/((2k+1)\pi))^n$; $z = \infty$ is a non-isolated singularity.

7.10. The *Bernoulli numbers* B_n are defined by the power series of $z/(e^z-1)$:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n .$$

Prove that Bernoulli numbers satisfy:

$$a)\ \binom{n+1}{0} B_0 + \binom{n+1}{1} B_1 + \cdots + \binom{n+1}{n} B_n = 0\,, \qquad \text{for } n \geq 1$$

$$b)\ B_0 = 1\,,\ B_1 = -\frac{1}{2}\,,\ B_2 = \frac{1}{6}\,,\ B_4 = -\frac{1}{30}\,,\ B_6 = \frac{1}{42}\,,\ B_8 = -\frac{1}{30}\,,$$

$$c)\ B_{2n+1} = 0\,, \qquad \text{for } n \geq 1\,.$$

Hint: Check first that the function $z/2 + z/(e^z-1)$ is even.

d) Prove that $\limsup_{n\to\infty} \left(\frac{|B_n|}{n!}\right)^{1/n} = \frac{1}{2\pi}$.

Hint: Find the distance from 0 to the closest singularity of the function $z/(e^z-1)$.

7.11. Compute the power series about 0 of each of the following functions, by applying the previous exercise, and find the radius of convergence in each case:

$$a)\ \frac{z}{e^z+1}\,, \qquad \left(\text{prove first that } \frac{z}{e^z+1} = \frac{z}{e^z-1} - \frac{2z}{e^{2z}+1}\right)$$

$$b)\ \tan z\,, \qquad \left(\text{prove first that } \tan z = -i + \frac{2i}{e^{2iz}+1}\right)$$

$$c)\ z \operatorname{cotan} z\,, \qquad \left(\text{prove first that } z \operatorname{cotan} z = iz + \frac{2iz}{e^{2iz}-1}\right)$$

$$d)\ \frac{z}{\sin z}\,, \qquad \left(\text{prove first that } \frac{z}{\sin z} = \frac{iz}{e^{iz}-1} + \frac{iz}{e^{iz}+1}\right)$$

$$e)\ \log\cos z\,, \qquad \left(\text{prove first that } (\log\cos z)' = -\tan z\right)$$

$$f)\ \log\frac{\tan z}{z}\,, \qquad \left(\text{prove first that } \left(\log\frac{\tan z}{z}\right)' = 2\operatorname{cosec}(2z) - \frac{1}{z}\right)$$

$$g)\ \log\frac{z}{\sin z}\,, \qquad \left(\text{prove first that } \left(\log\frac{z}{\sin z}\right)' = \frac{1}{z} - \operatorname{cotan} z\right)$$

Solutions: a) $\sum_{n=1}^{\infty} B_n(1-2^n)z^n/n!$, $R=\pi$,
b) $\sum_{n=1}^{\infty}(-1)^n B_{2n}(1-4^n)4^n z^{2n-1}/(2n)!$, $R=\pi/2$,
c) $\sum_{n=0}^{\infty}(-1)^n B_{2n}4^n z^{2n}/(2n)!$, $R=\pi$,
d) $\sum_{n=0}^{\infty}(-1)^{n+1}(2^{2n}-2)B_{2n}z^{2n}/(2n)!$, $R=\pi$,
e) $\sum_{n=1}^{\infty}(-1)^{n+1}B_{2n}(1-4^n)4^n z^{2n}/[2n\,(2n)!]$, $R=\pi/2$,
f) $\sum_{n=0}^{\infty}(-1)^{n+1}2^{2n}(2^{2n}-2)B_{2n}z^{2n}/[2n(2n)!]$, $R=\pi/2$,
g) $\sum_{n=1}^{\infty}(-1)^{n+1}4^n B_{2n}z^{2n}/[2n\,(2n)!]$, $R=\pi$.