COMPLEX VARIABLE AND TRANSFORMS | TEST 2

Problem 1 (1) Compute the Fourier series of the function $f(x) = x$ defined on $[-L, L]$.
(2) At which points x of the interval $[-L, L]$ does the Fourier series of f converge to $f(x)$?.
(3) Compute the Fourier sine series of the function $f(x) = x$ defined on $[0, L]$.

---

(1) Since $f(x) = x$ and $\sin \frac{n\pi x}{L}$ are odd functions, and $\cos \frac{n\pi x}{L}$ are even functions, we have

$$a_n = \frac{1}{L}\int_{-L}^{L} x \cos \frac{n\pi x}{L}\, dx = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{L} x \sin \frac{n\pi x}{L}\, dx = \left[\frac{-2}{n\pi}\, x \cos \frac{n\pi x}{L}\right]_{x=0}^{x=L} + \frac{2}{n\pi}\int_{0}^{L} \cos \frac{n\pi x}{L}\, dx$$

$$= \frac{-2L}{n\pi} \cos n\pi + \left[\frac{2L}{n^2\pi^2} \sin \frac{n\pi x}{L}\right]_{x=0}^{x=L} = (-1)^{n+1}\,\frac{2L}{n\pi}\,.$$

Hence, the Fourier series of the function $f(x) = x$ on $[-L, L]$ is

$$\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{2L}{n\pi}\,\sin \frac{n\pi x}{L}\,.$$

(2) Since $f(x) = x$ is piecewise continuously differentiable on $[-L, L]$, then the Fourier series of $f(x)$ converges:
• to the periodic extension of f, at those points x for which the periodic extension of f is continuous,
• to the average of the two side limits of the periodic extension of f

$$\frac{1}{2}\big(f(x+) + f(x-)\big),$$

at those points x for which the periodic extension of f has a jump discontinuity (here, the periodic extension of f is denoted by f also).
Hence, this Fourier series converges to $f(x) = x$ for every $x \in (-L, L)$, to

$$\frac{1}{2}\big(f(L+) + f(L-)\big) = \frac{1}{2}\big(L - L\big) = 0 \neq L = f(L),$$

if $x = L$, and to

$$\frac{1}{2}\big(f(-L+) + f(-L-)\big) = \frac{1}{2}\big(-L + L\big) = 0 \neq -L = f(-L),$$

if $x = -L$. Therefore, this Fourier series converges to $f(x) = x$ if and only if $x \in (-L, L)$.

(3) Since $f(x) = x$ is an odd function, its Fourier sine series on $[0, L]$ is its Fourier series on $[-L, L]$:

$$\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{2L}{n\pi}\,\sin \frac{n\pi x}{L}\,.$$

Problem 2 Find a solution of the initial value problem for the diffusion equation with absorption:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = k\,\frac{\partial^2}{\partial x^2}u(x,t) - c\,u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Denote by $U(\omega, t)$ and $F(\omega)$ the Fourier transforms on the variable x of the functions $u(x,t)$ and $f(x)$, respectively. If we apply the Fourier transform on the variable x to the partial differential equation, we obtain

$$\begin{cases} \frac{\partial}{\partial t}U(\omega,t) = -k\,\omega^2 U(\omega,t) - c\,U(\omega,t)\,, \\ U(\omega,0) = F(\omega)\,. \end{cases}$$

For each fixed ω, we have the ordinary differential equation $\frac{\partial}{\partial t}U(\omega,t) = -(k\omega^2 + c)U(\omega,t)$, whose general solution is $U(\omega,t) = A\,e^{-(k\omega^2+c)t}$, where A is a constant (with respect to the variable t, and so A can depend on the variable ω). By substituting the initial condition $U(\omega,0) = F(\omega)$ we obtain $A = F(\omega)$ and so, $U(\omega,t) = e^{-ct}F(\omega)\,e^{-k\omega^2 t}$. Let us define the function $K_t(x)$ by

$$K_t(x) = \sqrt{\frac{\pi}{kt}}\;e^{-x^2/(4kt)}, \qquad \text{with} \qquad \mathcal{F}\big[K_t\big](\omega) \;=\; e^{-k\omega^2 t}.$$

Then,

$$\mathcal{F}\big[u\big](\omega) = e^{-ct}\mathcal{F}\big[K_t\big](\omega)\,\mathcal{F}\big[f\big](\omega) = e^{-ct}\mathcal{F}\big[K_t * f\big](\omega)\,,$$

$$u(x,t) = e^{-ct}(K_t * f)(x) = \frac{e^{-ct}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\,f(y)\;dy\,.$$

Problem 3 Find a function satisfying $f(0) = 1$ and

$$f'(x) = 1 - \int_0^x f(x-s)\,e^{-2s}\,ds\,,$$

by using the Laplace transform.

Denote by $F(z)$ the Laplace transforms of the function $f(x)$. Note that the equation can be written as: $f' = 1 - e^{-2x} * f$. By applying the Laplace transform to this equation, we obtain

$$\begin{aligned} zF(z) - 1 &= \frac{1}{z} - \frac{1}{z+2}\,F(z), \\ zF(z) + \frac{1}{z+2}\,F(z) &= \frac{1}{z} + 1, \\ \frac{z^2+2z+1}{z+2}\,F(z) &= \frac{z+1}{z}\,, \\ F(z) &= \frac{(z+2)(z+1)}{z(z+1)^2} = \frac{z+2}{z(z+1)}\,, \\ F(z) &= \frac{2}{z} - \frac{1}{z+1}\,, \\ f(x) &= 2 - e^{-x}. \end{aligned}$$

Problem 4 Solve the following difference equation by using the Z-transform,

$$f_{n+2} - f_{n+1} - 2f_n = 0, \qquad \text{with} \qquad f_0 = 0,\ f_1 = 1.$$

By applying the Z-transform to the difference equation, if $F(z) = (Zf_n)(z)$, we obtain

$$\begin{aligned}
&z^2\Big(F(z) - \frac{1}{z}\Big) - zF(z) - 2F(z) = 0,\\
&(z^2 - z - 2)F(z) = z,\\
&F(z) = z\,\frac{1}{(z+1)(z-2)} = z\Big(\frac{1}{3}\,\frac{1}{z-2} - \frac{1}{3}\,\frac{1}{z+1}\Big),\\
&f_n = \frac{1}{3}\,2^n - \frac{1}{3}\,(-1)^n.
\end{aligned}$$

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García