COMPLEX VARIABLE AND TRANSFORMS | TEST 3 - SOLUTIONS

Problem 1 Find the radius of convergence of the series (without using Stirling formula):

$$(1) \quad \sum_{n=1}^{\infty} n z^{n^3} = z + 2z^8 + 3z^{27} + \dots, \qquad (2) \quad \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+3)!} z^n.$$

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(1) We have $a_k = n$ if $k = n^3$ for some $n \geq 1$ and $a_k = 0$ otherwise. Thus, $a_{n^3} = n$ and

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{n\to\infty} |a_{n^3}|^{1/n^3} = \limsup_{n\to\infty} n^{1/n^3}.$$

Let us compute the limit

$$L = \lim_{t\to\infty} t^{1/t^3}$$

if it exists. By taking logarithms and applying Bernouilli-l'Hôpital's rule, we obtain

$$\log L = \log \lim_{t\to\infty} t^{1/t^3} = \lim_{t\to\infty} \log\left(t^{1/t^3}\right) = \lim_{t\to\infty} \frac{\log t}{t^3} = \lim_{t\to\infty} \frac{1/t}{3t^2} = \lim_{t\to\infty} \frac{1}{3t^3} = 0,$$

and so, the limit exists and $L = e^0 = 1$. Hence,

$$\frac{1}{R} = \limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{n\to\infty} |a_{n^3}|^{1/n^3} = \limsup_{n\to\infty} n^{1/n^3} = \lim_{t\to\infty} t^{1/t^3} = 1,$$

and so, $R = 1$.

(2) Since there exists the limit

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)!(n+4)!}}{\frac{(2n)!}{n!(n+3)!}} = \lim_{n\to\infty} \frac{(2n+2)!n!(n+3)!}{(2n)!(n+1)!(n+4)!}$$
$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+4)} = 4,$$

we have

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 4.$$

Hence, $R = 1/4$.

Problem 2 (1) Obtain the Laurent's series on the annulus $\{1 < |z| < 2\}$ of the function

$$f(z) = \frac{2z-1}{z^2 - z - 2}.$$

(2) Obtain the Laurent's series of the same function f on the annulus $\{2 < |z| < \infty\} = \{2 < |z|\}$.

If we write f as sum of simple fractions, we obtain

$$f(z) = \frac{1}{z+1} + \frac{1}{z-2}\,.$$

The formula of the sum of a geometric series gives

$$\frac{1}{z-2} = \frac{-1}{2}\,\frac{1}{1-z/2} = \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\,z^n,$$

which converges if $|z/2| < 1$, i.e., $|z| < 2$.
Also,

$$\frac{1}{z+1} = \frac{1}{z}\,\frac{1}{1-(-1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}\,,$$

which converges if $|-1/z| < 1$, i.e., if $1 < |z|$.
Hence,

$$f(z) = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\,z^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}\,,$$

if $1 < |z| < 2$.
We also have

$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-2/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}\,,$$

which converges if $|2/z| < 1$, i.e., if $2 < |z|$.
Therefore,

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{2^n+(-1)^n}{z^{n+1}}\,,$$

if $2 < |z|$.

Problem 3 (1) Compute the principal value of the following integral for $\omega > 0$, checking that the hypotheses (of the theorem that you use) hold:

$$p.v.\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{x(x^2+4)}\,dx.$$

(2) Compute this principal value for $\omega < 0$ by using the previous item.

The function $z(z^2+4)$ has simple poles at $z = 0$ and $z = \pm 2i$ (since $z(z^2+4)$ is a polynomial with degree 3); $z = 0$ is the only pole with zero imaginary part (it is in the real line), $z = 2i$

is the only pole with positive imaginary part (it is in the upper halfplane). Since $\omega > 0$, $1/(z^3+4z)$ is holomorphic on $\mathbb{C} \setminus \{0, \pm 2i\}$ and $\lim_{z\to\infty} 1/(z^3+4z) = 0$, we have

$$p.v. \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x(x^2+4)}\,dx = 2\pi i \text{ Res}\left(\frac{e^{i\omega z}}{z(z^2+4)}, 2i\right) + \pi i \text{ Res}\left(\frac{e^{i\omega z}}{z(z^2+4)}, 0\right)$$
$$= 2\pi i \lim_{z\to 2i} \frac{(z-2i)\,e^{i\omega z}}{z(z-2i)(z+2i)} + \pi i \lim_{z\to 0} \frac{z\,e^{i\omega z}}{z(z^2+4)}$$
$$= 2\pi i\,\frac{e^{-2\omega}}{-8} + \pi i\,\frac{1}{4} = \frac{\pi i\big(1-e^{-2\omega}\big)}{4}\,.$$

If $\omega < 0$, then $-\omega > 0$ and the previous item gives

$$p.v. \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x(x^2+4)}\,dx = p.v. \int_{-\infty}^{\infty} \frac{\overline{e^{i(-\omega)x}}}{x(x^2+4)}\,dx = \overline{p.v. \int_{-\infty}^{\infty} \frac{e^{i(-\omega)x}}{x(x^2+4)}\,dx}$$
$$= \overline{\frac{\pi i\big(1-e^{-2(-\omega)}\big)}{4}} = \frac{\pi i\big(e^{2\omega}-1\big)}{4}\,.$$

Problem 4 Find a solution of the initial value problem for the heat equation:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; \frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) \;=\; e^{-x^2}, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Applying the Fourier transform in the variable x to the PDE we obtain:

$$u_t = u_{xx} \quad \Longrightarrow \quad \hat{u}_t = -\omega^2 \hat{u}\,.$$

For each fixed $\omega \in \mathbb{R}$ this last equation is an ordinary differential equation on the variable t whose solution is $\hat{u}(\omega,t) = C(\omega)\,e^{-\omega^2 t}$ where $\omega \in \mathbb{R}$ and $t > 0$. From this equation we deduce that $C(\omega) = \hat{u}(\omega,0)$, but using now the initial condition $u(x,0) = e^{-x^2}$ we deduce taking again Fourier transforms that $C(\omega) = \hat{u}(\omega,0) = \mathcal{F}[e^{-x^2}](\omega) = \frac{1}{\sqrt{4\pi}}\,e^{-\omega^2/4}$.
Hence,

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{4\pi}}\,e^{-\omega^2/4}e^{-\omega^2 t} = \frac{1}{\sqrt{4\pi}}\,e^{-\omega^2(t+1/4)}$$

and taking now the inverse Fourier transform, we obtain that

$$u(x,t) = \frac{1}{\sqrt{4\pi}}\mathcal{F}^{-1}[e^{-\omega^2(t+1/4)}](x) = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{\pi}{t+1/4}}\,e^{-x^2/[4(t+1/4)]} = \frac{1}{\sqrt{4t+1}}\,e^{-x^2/(4t+1)}\,.$$

Problem 5
Solve the following equation

$$f(t) = \cos t + \int_0^t e^{-s} f(t-s)ds$$

by using the Laplace transform.

Applying the Laplace transform we get

$$F(s) = \frac{s}{s^2+1} + \frac{F(s)}{s+1}$$

where $F(s)$ is the Laplace transform of f. Then,

$$F(s)\frac{s}{s+1} = \frac{s}{s^2+1}.$$

So,

$$F(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

and applying the inverse Laplace transform

$$f(t) = \cos t + \sin t.$$

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