uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 1: Measure theory

Section 1.1: Measurable spaces

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1 Measure Theory

1.1. Measurable spaces

1.1.1 Measurability: Topological spaces versus Measurable spaces

Definition 1.1 A collection $\mathcal{T}$ of subsets of a set X is said to be a *topology* on X and also $(X, \mathcal{T})$ is said to be a *topological space*, if

(a) $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$.

(b) If $V_1, V_2, \dots, V_n \in \mathcal{T}$ then $V_1 \cap V_2 \cap \dots \cap V_n \in \mathcal{T}$.

(c) If $\{V_\alpha\}_{\alpha \in A}$ is an arbitrary collection of members of $\mathcal{T}$, then $\cup_{\alpha \in A} V_\alpha \in \mathcal{T}$.

The members of $\mathcal{T}$ are called *open sets.*

Example 1.2 (1) $X = \bar{\mathbb{R}} = [-\infty, \infty]$; the open sets are (a, b), $[-\infty, a)$, $(b, \infty]$ and any union of sets of these types.
(2) Given a *metric space*, i.e. a set X with a *distance* function $d : X \times X \longrightarrow [0, \infty)$ verifying:

- $d(x, y) = 0 \Leftrightarrow x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Given $x \in X$, $r > 0$, the *open ball with center x and radius r* is: $B(x, r) = \{x \in X : \; d(x, y) < r\}$.
The collection of all sets that are arbitrary unions of open balls is a topology on X.
(3) For example $X = \mathbb{R} = (-\infty, \infty)$ is a metric space with the distance $d(x, y) = |x - y|$. The open balls are the intervals (a, b) and so the open sets are the arbitrary unions of intervals (a, b).

Definition 1.3 (Global continuity) Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be topological spaces and let $f : X \longrightarrow Y$ be a mapping. We say that f is *continuous* if $f^{-1}(V) \in \mathcal{T}$ for all $V \in \mathcal{T}'$.

A *neighborhood* of a point in a topological space is an open set containing it.

Definition 1.4 (Local continuity) f is *continuous at $x_0 \in X$* if for all neighborhood V of $f(x_0)$ in Y, there exist a neighborhood W of x_0 in X with $f(W) \subseteq V$.

For metric spaces this definition of (local) continuity is equivalent to the usual $\varepsilon - \delta$ definition.

Proposition 1.5 *Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be topological spaces. Then, a mapping $f : X \longrightarrow Y$ is continuous if and only if f is continuous at every $x \in X$.*

Definition 1.6 A collection $\mathcal{A}$ of subsets of a set X is said to be a *σ-algebra* on X, if

(a) $\varnothing \in \mathcal{A}$.

(b) If $A \in \mathcal{A}$, then $A^c = X \setminus A \in \mathcal{A}$.

(c) If $\{A_j\}_{j \in \mathbb{N}}$ is a countable collection of members of $\mathcal{A}$, then $\cup_{j=1}^{\infty} A_j \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a *measurable space* and the members of $\mathcal{A}$ are called *measurable sets.*
If property (c) is verified only for finite unions, then $\mathcal{A}$ is called an *algebra*.

Example 1.7 Consider any set X.
(1) The *power set* $\mathcal{P}(X)$ of X (the set of all subsets of X) is a σ-algebra on X.
(2) $\{\varnothing, X\}$ is a σ-algebra on X.

Definition 1.8 Let $(Y, \mathcal{T})$ be a topological space, $(X, \mathcal{A})$ be a measurable space and let $f : X \longrightarrow Y$ be a mapping. We say that f is <u>measurable</u> if $f^{-1}(V) \in \mathcal{A}$ for all $V \in \mathcal{T}$.

Proposition 1.9 *Let $(X, \mathcal{A})$ be a measurable space. Then*

i) $X \in \mathcal{A}$.
ii) $A_1, \ldots, A_n \in \mathcal{A} \implies \cup_{j=1}^{n} A_j \in \mathcal{A}$.
iii) $\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{A} \implies \cap_{j=1}^{\infty} A_j \in \mathcal{A}$.
iv) $A, B \in \mathcal{A} \implies A \setminus B \in \mathcal{A}$.

Proposition 1.10 *Let $\mathcal{S} \subset \mathcal{P}(X)$. Then*

$$\sigma(\mathcal{S}) = \mathcal{A}_{\mathcal{S}} = \bigcap \{\mathcal{A} : \ \mathcal{A} \textit{ is } \sigma\textit{-algebra}, \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}$$

is a σ-algebra. It is called the <u>σ-algebra generated by $\mathcal{S}$</u>.

Special Case: If $(X, \mathcal{T})$ is a topological space, the σ-algebra generated by $\mathcal{T}$, i.e. by the open sets, is called the <u>*Borel σ-algebra*</u> $\mathcal{B}(X)$ and its members are called the <u>*Borel sets*</u>. Examples of Borel sets are open sets, closed sets and unions and intersections of a countable number of open or closed sets.

Proposition 1.11 *Let $(Y, \mathcal{T}_Y)$, $(Z, \mathcal{T}_Z)$ be topological spaces and let $g : Y \longrightarrow Z$ be a continuous mapping.*

i) If $(X, \mathcal{T}_X)$ is a topological space and $f : X \longrightarrow Y$ is continuous, then $h = g \circ f$ is continuous.

ii) If $(X, \mathcal{A})$ is a measurable space and $f : X \longrightarrow Y$ is measurable, then $h = g \circ f$ is measurable.

Proposition 1.12 *Let $u, v : X \longrightarrow \mathbb{R}$ be real measurable functions on a measurable space $(X, \mathcal{A})$. Let $\Phi : Imag(u, v) \subseteq \mathbb{R}^2 \longrightarrow Y$ be a continuous mapping into $(Y, \mathcal{T})$ topological space. Then $h = \Phi(u, v) : X \longrightarrow Y$ is measurable.*

Corollary 1.13

- *Let $u, v : X \longrightarrow \mathbb{R}$ be measurable functions, $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then: $u + v$, uv, $|u|^{\alpha}$ $(\alpha > 0)$, $\varphi \circ u$, $1/u$ (if $u(x) \neq 0$ for all $x \in X$) are measurable functions.*
- *If $f = u + iv$ and $u, v : X \longrightarrow \mathbb{R}$ are measurable, then $f : X \longrightarrow \mathbb{C}$ is measurable.*
- *If $f = u + iv$ is measurable with $u, v : X \longrightarrow \mathbb{R}$, then $u, v, |f|$ are real measurable functions.*
- *The first part also holds if we replace $\mathbb{R}$ by $\mathbb{C}$.*

1.1.2. Upper and lower limits

Definition 1.14 Let $\{a_n\}$ be a sequence in $\bar{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Then the sequence $b_k := \sup\{a_k, a_{k+1}, \ldots\}$ is monotonically decreasing: $b_1 \geq b_2 \geq \cdots$. Therefore:

$$\exists \, \beta := \lim_{n \to \infty} b_n = \inf_{n \in \mathbb{N}} b_n \, .$$

We call β the <u>*upper limit of the sequence* $\{a_n\}$</u> and write

$$\beta = \limsup_{n \to \infty} a_n = \lim_{k \to \infty} \sup\{a_j : \ j \geq k\} \, .$$

The <u>*lower limit*</u> is defined similarly (only interchange the symbols sup, inf in the above definition).

• The upper limit is the largest number which is limit of a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ and the lower limit is the smallest number with this property. Both numbers always exist.
• It is easy to see that if $\{a_n\}$ converges then

$$\limsup_{n \to \infty} a_n = \lim_{n \to \infty} a_n = \liminf_{n \to \infty} a_n \, .$$

Definition 1.15 Let $\{f_n\}$ be a sequence of functions $f_n : X \longrightarrow \bar{\mathbb{R}}$. Then, $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ are the functions:

$$(\sup_n f_n)(x) = \sup_n f_n(x)\,, \qquad (\limsup_{n\to\infty} f_n)(x) = \limsup_{n\to\infty} f_n(x)\,, \qquad \forall x \in X\,,$$
$$(\inf_n f_n)(x) = \inf_n f_n(x)\,, \qquad (\liminf_{n\to\infty} f_n)(x) = \liminf_{n\to\infty} f_n(x)\,, \qquad \forall x \in X\,.$$

If there exists $\lim_{n\to\infty} f_n(x)$ we define the function $f = \lim_{n\to\infty} f_n$ on the set of points where the convergence holds and we say that f is the *pointwise limit* of f_n on that set.

Theorem 1.16 *If $f_n : X \longrightarrow \bar{\mathbb{R}}$ are measurable functions for $n = 1, 2, 3, \ldots$, then*

$$\sup_n f_n,\ \inf_n f_n,\ \limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$$

are measurable functions. As a corollary, the limit $\lim_{n\to\infty} f_n$, if there exists, is a measurable function.

Corollary 1.17 *If $f, g : X \longrightarrow \bar{\mathbb{R}}$ are measurable functions, then $\max\{f, g\}$ and $\min\{f, g\}$ are also measurable functions. In particular*

$$f^+ = \max\{f, 0\}\,, \qquad f^- = -\min\{f, 0\}$$

are positive measurable functions. Let us observe that $f = f^+ - f^-$, $|f| = f^+ + f^-$.