uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 2: Integration theory

Section 2.1: Integration of positive functions

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

2 Integration theory

2.1. Integration of positive functions

2.1.1. Simple functions

A *simple function* on a measure space $(X, \mathcal{A}, \mu)$ is a measurable function whose range consists of only finitely many points. In other words, a simple function $s : X \longrightarrow \mathbb{R}$ is given by

$$s = \sum_{j=1}^{n} c_j \, \chi_{A_j} \,, \qquad \text{with } c_j \in \mathbb{R},\ A_j \in \mathcal{A}\,.$$

Theorem 2.1 *Let $f : X \longrightarrow [0, \infty]$ be a positive measurable function. Then, there exists simple positive measurable functions $\{s_n\}_{n=1}^{\infty}$ such that*

(1) $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq \cdots \leq f$.

(2) $s_n(x) \to f(x)$ *as* $n \to \infty$, $\forall\, x \in X$.

Besides, if f is bounded then $s_n \to f$ uniformly.

This crucial result gives as a method to prove results about measurable functions: 1) to prove it for characteristic functions; 2) to prove it for simple functions; 3) to prove it for positive functions by passing to the limit; 4) to prove it for real functions using that $f = f^+ - f^-$. For example:

Corollary 2.2 *Let $f, g : X \longrightarrow \mathbb{R}$ be measurable functions and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). Then λf, $f + g$, fg and $1/f$ are measurable functions.*

2.1.2. Positive functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a positive bounded function: $0 \leq f \leq H$. Let us do a partition of the range of f: $[0, H] = \cup_j [c_{j-1}, c_j)$. Then, it is easy to check that

$$s := \sum_j c_{j-1} \, \chi_{f^{-1}([c_{j-1}, c_j))} \leq f \leq t := \sum_j c_j \, \chi_{f^{-1}([c_{j-1}, c_j))} \,,$$

and then it has sense to write:

$$\int_X s \, d\mu := \sum_j c_{j-1} \, \mu(f^{-1}([c_{j-1}, c_j))) \leq \int_X f \, d\mu \leq \sum_j c_j \, \mu(f^{-1}([c_{j-1}, c_j))) =: \int_X t \, d\mu \,.$$

Therefore we see that we could approximate the integral $\int_X f$ by upper and lower sums like in the Riemann integral, but now the integral is more general because we are using measurable sets instead of only intervals (in the Riemann integral we approximate by simple functions of type $\sum_j c_j \, \chi_{I_j}$ with I_j intervals. As we will see, this approach will allow us to obtain good convergence results.

Definition 2.3 Let $s = \sum_{j=1}^{n} c_j \, \chi_{A_j}$ be a measurable simple function. We define the *integral of s* as

$$\int_X s = \int_X s \, d\mu := \sum_{j=1}^{n} c_j \, \mu(A_j)$$

and if $E \in \mathcal{A}$:

$$\int_E s = \int_E s \, d\mu := \sum_{j=1}^{n} c_j \, \mu(A_j \cap E) \,.$$

Now, if $f : X \longrightarrow [0,\infty]$ is a positive measurable function, we define its *integral* as

$$\int_E f = \int_E f \, d\mu := \sup_{0 \le s \le f} \int_E s$$

where the supremum is extended over all the simple functions s such that $0 \le s \le f$.

Therefore, every measurable positive function has Lebesgue integral. The value of the integral can be zero, positive or infinite. We say that f is *Lebesgue-integrable* if $\int_X f < \infty$.

Properties of the Lebesgue integral.
Let $f, g \ge 0$ be measurable functions, $A, B, E \in \mathcal{A}$, $\lambda \ge 0$. Then

(1) $\int_E f = \int_X f \chi_E$.

(2) $\int_E (f+g) = \int_E f + \int_E g$.

(3) $\int_E \lambda f = \lambda \int_E f$.

(4) $f \le g \implies \int_E f \le \int_E g$.

(5) $A \subseteq B \implies \int_A f \le \int_B f$.

(6) $\int_E f = 0 \iff f = 0$ almost everywhere on E, i.e. $\mu(\{x \in E : f(x) \ne 0\}) = 0$.

(7) $\mu(E) = 0 \implies \int_E f = 0$.

(8) $A \cap B = \varnothing \implies \int_{A \cup B} f = \int_A f + \int_B f$.

In fact, we have more:

Proposition 2.4 *Let $f, g \ge 0$ be measurable functions. Then*

$$f \le g \ \text{a.e.} \implies \int_X f \le \int_X g\,.$$

In particular, if $f = g$ a.e. then $\int_X f = \int_X g$.

2.1.3. Lebesgue's monotone convergence theorem and Fatou's lemma

Theorem 2.5 (Monotone convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that*

$$0 \le f_1(x) \le f_2(x) \le \cdots \le \infty\,, \qquad \forall\, x \in X\,.$$

Then

$$\int_X \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_X f_n\,.$$

This result has a lot of consequences. For example:

Corollary 2.6 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of positive measurable functions and*

$$f(x) := \sum_{n=1}^{\infty} f_n(x) \qquad x \in X\,.$$

Then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu\,.$$

Corollary 2.7 *If $a_{ij} \geq 0$ for $i, j \in \mathbb{N}$, then:* $\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

Theorem 2.8 (Fatou's lemma).
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of positive measurable functions. Then

$$\int_X \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_X f_n \,.$$

2.1.4. Measures defined by a density

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \longrightarrow [0, \infty]$ be a positive measurable function. Let us define

$$\varphi(E) := \int_E f \, d\mu \,, \qquad \forall\, E \in \mathcal{A} \,.$$

Then, φ is a measure on $\mathcal{A}$, and

$$\int_X g \, d\varphi = \int_X gf \, d\mu \,, \qquad \forall\, g : X \longrightarrow [0, \infty] \text{ measurable.}$$

This fact justifies the notation $d\varphi = f \, d\mu$. This measure is said to be defined by the density function f.