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Departamento de Matemáticas

Integration and Measure

Chapter 2: Integration theory

Section 2.3: Integration on product spaces

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

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2.3. Integration on product spaces

Along this section $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ will denote σ-finite measure spaces. Let us consider the product measure space $(X \times Y, \overline{\mathcal{A} \otimes \mathcal{B}}, \mu \otimes \nu)$.

Notations.
If $E \subseteq X \times Y$ and $x \in X$, we define the *section* E_x of E as

$$E_x := \{y \in Y : \ (x, y) \in E\} \subseteq Y \, ,$$

and if $y \in Y$, the section E^y of E as

$$E^y := \{x \in X : \ (x, y) \in E\} \subseteq X \, .$$

If $f : X \times Y \longrightarrow \bar{\mathbb{R}}$, given $x \in X$, the section f_x of f is the function $f_x : Y \longrightarrow \bar{\mathbb{R}}$ given by $f_x(y) = f(x, y)$, and given $y \in Y$, the section f^y of f is the function $f^y : X \longrightarrow \bar{\mathbb{R}}$ given by $f^y(x) = f(x, y)$.

Proposition 2.1 (1) *If $E \in \mathcal{A} \otimes \mathcal{B}$ then $E_x \in \mathcal{B}$ for all $x \in X$ and $E^y \in \mathcal{A}$ for all $y \in Y$.*

(2) *If $f : X \times Y \longrightarrow \bar{\mathbb{R}}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable then f_x is $\mathcal{B}$-measurable for all $x \in X$ and f^y is $\mathcal{A}$-measurable for all $y \in Y$.*

Proposition 2.2 (Cavalieri's principle: Volume calculus by sections) *Let $E \in \mathcal{A} \otimes \mathcal{B}$. Then*

(1) *The function $g(x) = \nu(E_x)$ is $\mathcal{A}$-measurable and*

$$\int_X g \, d\mu = \int_X \nu(E_x) \, d\mu = (\mu \otimes \nu)(E) \, .$$

(2) *The function $h(y) = \mu(E^y)$ is $\mathcal{B}$-measurable and*

$$\int_Y h \, d\nu = \int_Y \mu(E^y) \, d\nu = (\mu \otimes \nu)(E) \, .$$

Theorem 2.3 (Tonelli-Fubini theorem) *Let $f : X \times Y \longrightarrow [0, \infty]$ be a positive $\mathcal{A} \otimes \mathcal{B}$-measurable function. Then*

(1) *The function $F(x) = \int_Y f_x \, d\nu$ is $\mathcal{A}$-measurable and*

$$\int_X F \, d\mu = \int_{X \times Y} f \, d(\mu \otimes \nu) \, .$$

(2) *The function $G(y) = \int_X f^y \, d\mu$ is $\mathcal{B}$-measurable and*

$$\int_Y G \, d\nu = \int_{X \times Y} f \, d(\mu \otimes \nu) \, .$$

Therefore,

$$\int_{X \times Y} f \, d(\mu \otimes \nu) = \int_X d\mu(x) \int_Y f(x, y) \, d\nu(y) = \int_Y d\nu(y) \int_X f(x, y) \, d\mu(x) \, .$$

Theorem 2.4 (Fubini's theorem) *Let $f : X \times Y \longrightarrow \bar{\mathbb{R}}$ be an $\mathcal{A} \otimes \mathcal{B}$-measurable function. Then the integrals*

$$I_1(f) = \int_{X \times Y} |f(x, y)| \, d\mu(x) \, d\nu(y)$$

and

$$I_2(f) = \int_X d\mu(x) \int_Y |f(x, y)| \, d\nu(y) \, , \qquad I_3(f) = \int_Y d\nu(y) \int_X |f(x, y)| \, d\mu(x) \, ,$$

exist and are equal (they can be finite or infinite). Besides, if they are finite, (i.e. if $f \in L^1(\mu \otimes \nu)$) then

$$\int_{X \times Y} f(x, y) \, d\mu(x) \, d\nu(y) = \int_X d\mu(x) \int_Y f(x, y) \, d\nu(y) = \int_Y d\nu(y) \int_X f(x, y) \, d\mu(x) \, .$$

2.3.1. Integration on $\mathbb{R}^n$ using polar coordinates

Given $x \in \mathbb{R}^n \setminus \{0\}$, let us consider its polar coordinates (r, x') where $r = \|x\| \in (0, \infty)$, $x' = x/\|x\| \in S_{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The mapping

$$\varphi : \mathbb{R}^n \setminus \{0\} \longrightarrow (0, \infty) \times S_{n-1} \qquad \text{given by } \varphi(x) = (r, x')$$

is a bijection. We have that:

a) If μ is the image measure under φ of the Lebesgue measure on $\mathbb{R}^n \setminus \{0\}$, then

$$\mu(E \times U) = \sigma(U) \int_E r^{n-1} dr\,, \qquad \text{for all Borel sets } E \subseteq (0, \infty),\ U \subseteq S_{n-1}\,.$$

b) If $f : \mathbb{R}^n \setminus \{0\} \longrightarrow [0, \infty]$ is a positive measurable function, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty r^{n-1} dr \int_{S_{n-1}} f(rx')\, d\sigma(x')$$

where σ is the $(n-1)$-dimensional Lebesgue measure on S_{n-1}.