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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 1: Measure theory

Section 1.1: Measurable spaces

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1 Measure Theory

1.1 Measurable spaces

Problem 1.1.1 Let $f : X \longrightarrow Y$ be a mapping. Given a subset $A \subseteq Y$ let us define:

$$f^{-1}(A) = \{x \in X : \ f(x) \in A\}.$$

Prove that

i) $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

ii) $f^{-1}(\bigcup_j A_j) = \bigcup_j f^{-1}(A_j)$.

iii) $f^{-1}(\bigcap_j A_j) = \bigcap_j f^{-1}(A_j)$.

Hint: To prove that two sets A and B are equal you must prove that each element belonging to A also belongs to B and reciprocally each element in B also belongs to A.
Solution: i) $x \in f^{-1}(Y \setminus A) \iff f(x) \in Y \setminus A \iff f(x) \notin A \iff x \notin f^{-1}(A) \iff x \in X \setminus f^{-1}(A)$.

Problem 1.1.2 Let $f : X \longrightarrow Y$ be a mapping between two topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{T}')$. Prove that f is continuous if and only if f is continuous at every $x \in X$.

Hint: To prove an statement of type $A \iff B$ you must prove that if we assume that A holds, then B also holds and viceversa.
Solution: ($\Rightarrow$) If f is continuous and $V \in \mathcal{T}'$ verifies $f(x_0) \in V$ then $x_0 \in f^{-1}(V) \in \mathcal{T}$ and so $W := f^{-1}(V) \in \mathcal{T}$ satisfies $f(W) \subseteq V$, that is to say: f is continuous at x_0.
($\Leftarrow$) Let $V \in \mathcal{T}'$ and $x_0 \in f^{-1}(V)$. Then $f(x_0) \in V$ and, as f is continuous at x_0, $\exists W_{x_0} \in \mathcal{T}$ such that $f(W_{x_0}) \subseteq V$. Hence, $W_{x_0} \subseteq f^{-1}(V)$ and $f^{-1}(V) = \cup_{x_0 \in f^{-1}(V)} W_{x_0}$. Therefore, as $W_{x_0} \in \mathcal{T}$, we have that $f^{-1}(V) \in \mathcal{T}$ and f is continuous.

Problem 1.1.3

i) Show that if $X = \{1, 2, 3\}$, then $\mathcal{F} := \{\emptyset, \{2, 3\}, X\}$ is not a σ-algebra.

ii) Let $X = \{a, b, c, d\}$. Check that the family of subsets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$

is a σ-algebra in X.

Hint: You must check if the properties of a σ-algebra are satisfied.
Solution: i) $X \setminus \{2, 3\} = \{1\} \notin \mathcal{F}$; ii) All properties of a σ-algebra are satisfied.

Problem 1.1.4 Let $\mathcal{S}$ be a family of subsets of X, $\mathcal{S} \subseteq \mathcal{P}(X)$. Prove that

$$\mathcal{A}_{\mathcal{S}} = \bigcap \{\mathcal{A} : \ \mathcal{A} \text{ is a } \sigma\text{-algebra}, \ \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}$$

is the smallest σ-algebra containing $\mathcal{S}$.

Note: $\mathcal{A}_{\mathcal{S}}$ is called the σ-algebra generated by $\mathcal{S}$ and sometimes is denoted as $\sigma(\mathcal{S})$.
Hint: Prove that $\mathcal{A}_{\mathcal{S}}$ is a σ-algebra.
Solution: $\mathcal{A}_{\mathcal{S}}$ is a σ-algebra since:

a) For every σ-algebra $\mathcal{A}$ we have $\varnothing \in \mathcal{A}$ and so $\varnothing \in \mathcal{A}_{\mathcal{S}}$.

b) If $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}_{\mathcal{S}}$ then $\cup_j A_j \in \mathcal{A}$ for all σ-algebra with $\mathcal{S} \subseteq \mathcal{A}$. Hence $\cup_j A_j \in \mathcal{A}_{\mathcal{S}}$.

c) If $A \in \mathcal{A}_{\mathcal{S}}$ then $A \in \mathcal{A}$ for every σ-algebra with $\mathcal{S} \subseteq \mathcal{A}$. Hence $X \setminus A \in \mathcal{A}$ for every σ-algebra with $\mathcal{S} \subseteq \mathcal{A}$ and so $X \setminus A \in \mathcal{A}_{\mathcal{S}}$.

Finally, by definition, $\mathcal{A}_{\mathcal{S}}$ is the smallest σ-algebra containing $\mathcal{S}$.

Problem 1.1.5 Let $X = \{a, b, c, d\}$. Construct the σ-algebra generated by

$$\mathcal{E}_1 = \{\{a\}\} \ \text{ y } \ \text{por} \ \ \mathcal{E}_2 = \{\{a\}, \{b\}\}\,.$$

Hint: To construct them you must add the necessary subsets so that the σ-algebra properties are verified.

Solution: $\mathcal{A}_{\mathcal{E}_1} = \{\varnothing, \{a\}, \{b, c, d\}, X\}$, $\mathcal{A}_{\mathcal{E}_2} = \{\varnothing, \{a\}, \{b\}, \{b, c, d\}, \{a, c, d\}, \{a, b\}, \{c, d\}, X\}$.

Problem 1.1.6 Show with an example that the union of two σ-algebras does not have to be a σ-álgebra.

Hint: It suffices to consider a three-point set X.

Solution: Take $X = \{1, 2, 3\}$, $\mathcal{A} = \{\varnothing, \{1\}, \{2, 3\}, X\}$ and $\mathcal{B} = \{\varnothing, \{2\}, \{1, 3\}, X\}$. Then $\mathcal{A} \cup \mathcal{B}$ is not a σ-algebra since $\{1\}, \{2\} \in \mathcal{A} \cup \mathcal{B}$ but $\{1\} \cup \{2\} = \{1, 2\} \notin \mathcal{A} \cup \mathcal{B}$.

Problem 1.1.7 Determine the σ-algebra generated by the collection of all finite subsets of a non-countable set X.

Solution: $\mathcal{A} = \{A \subseteq X : \ A \text{ is finite or countable or } X \setminus A \text{ is finite or countable}\}$, since a countable union of finite or countable sets is a countable set.

Problem 1.1.8 Consider the σ-algebra of borelian subsets in $\mathbb{R}$. Is the following true or false?: There is a subset A of $\mathbb{R}$ which is not measurable, but such that $B = \{x \in A : x \text{ is irrational}\}$ is measurable.

Hint: Consider the set $C = \{x \in A : x \text{ is rational}\}$.

Solution: As $\{x\}$ is a borelian for every $x \in \mathbb{R}$, we have that C is measurable since C is countable. As, by hypothesis, B is also measurable we have that $B \cup C = A$ is measurable. A contradiction. Therefore B cannot be measurable.

Problem 1.1.9 Let $(X, \mathcal{A})$ be a measurable space and $(Y, \mathcal{T})$ be a topological space. Let us consider a mapping $f : X \longrightarrow Y$. Prove that

i) The collection $\mathcal{A}' = \{E \subseteq Y : \ f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra in Y. $\mathcal{A}'$ is called the *image σ-algebra of* $\mathcal{A}$.

ii) If f is measurable, then $\mathcal{B}(Y) \subseteq \mathcal{A}'$. Equivalently, if E is a borel set in Y, then $f^{-1}(E) \in \mathcal{A}$ and so $E \in \mathcal{A}'$.

Hint: ii) Prove that $\mathcal{T} \subseteq \mathcal{A}'$.

Solution: i) We have that: a) $\varnothing \in \mathcal{A} \Rightarrow \varnothing = f^{-1}(\varnothing) \in \mathcal{A}'$; b) $E \in \mathcal{A}' \Rightarrow f^{-1}(E) \in \mathcal{A}$. As $\mathcal{A}$ is a σ-algebra, we have $f^{-1}(Y \setminus E) = X \setminus f^{-1}(E) \in \mathcal{A} \Rightarrow Y \setminus E \in \mathcal{A}'$; c) $\{E_j\} \subset \mathcal{A}' \Rightarrow f^{-1}(E_j) \in \mathcal{A}$ for all j. As $\mathcal{A}$ is a σ-algebra, we have $f^{-1}(\cup_j E_j) = \cup_j f^{-1}(E_j) \in \mathcal{A} \Rightarrow \cup_j E_j \in \mathcal{A}'$.
ii) Let $V \in \mathcal{T}$. Then, as f is measurable, $f^{-1}(V) \in \mathcal{A}$ and so $V \in \mathcal{A}'$. Therefore $\mathcal{T} \subseteq \mathcal{A}'$ and so the σ-algebra generated by $\mathcal{T}$ is contained in $\mathcal{A}'$. But this σ-algebra is precisely the family $\mathcal{B}(Y)$ of borelian subsets.

Problem 1.1.10 Let $g : X \to Y$ be a mapping. Let $\mathcal{A}$ be a σ-algebra in Y. Prove that $\mathcal{A}' = \{g^{-1}(E) : E \in \mathcal{A}\}$ is a σ-algebra in X. $\mathcal{A}'$ is called the *pre-image σ-algebra of $\mathcal{A}$*.

Solution: We have that: a) $\varnothing \in \mathcal{A} \Rightarrow \varnothing = g^{-1}(\varnothing) \in \mathcal{A}'$; b) $E \in \mathcal{A}' \Rightarrow E' = f^{-1}(E)$ with $E \in \mathcal{A}$. As $\mathcal{A}$ is a σ-algebra, we have $X \setminus E' = f^{-1}(Y \setminus E) \in \mathcal{A}'$ since $Y \setminus E \in \mathcal{A}$; c) $\{E'_j\} \subset \mathcal{A}' \Rightarrow E'_j = f^{-1}(E_j)$ with $E_j \in \mathcal{A}$ for all j. As $\mathcal{A}$ is a σ-algebra, we have $\cup_j E'_j = f^{-1}(\cup_j E_j) \in \mathcal{A}'$ since $\cup_j E_j \in \mathcal{A}$.

Problem 1.1.11 A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* if the following conditions hold:

(1) $\emptyset \in \mathcal{A}$,

(2) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

(3) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Prove that an algebra $\mathcal{A}$ in X is a σ-álgebra if and only if it is closed for increasing countable unions of sets, that is to say:

$$E_i \in \mathcal{A}, \quad E_1 \subset E_2 \subset \dots \qquad \Longrightarrow \qquad \bigcup_1^\infty E_i \in \mathcal{A}\,.$$

Solution: ($\Rightarrow$) It is obvious; ($\Leftarrow$) If $\{A_j\}_{j=1}^\infty$ is an arbitrary collection of sets in $\mathcal{A}$, let $E_j = A_1 \cup \cdots \cup A_j$ for every $j \in \mathbb{N}$. Then $E_j \in \mathcal{A}$ for all j since $\mathcal{A}$ is an algebra. Also $\{E_j\}_{j=1}^\infty$ is an increasing family since obviously $E_j \subseteq E_{j+1}$. Hence, $\cup_{j=1}^\infty E_j \in \mathcal{A}$ since $\mathcal{A}$ is closed for increasing union of sets. But $\cup_{j=1}^\infty A_j = \cup_{j=1}^\infty E_j$ and so $\cup_{j=1}^\infty A_j \in \mathcal{A}$.

Problem 1.1.12 Let $u, v : X \longrightarrow \mathbb{R}$ be measurable functions and let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Prove that

i) $\varphi \circ u$ is measurable.

ii) $u + v$, uv, $|u|^\alpha$ ($\alpha > 0$) are measurable functions.

iii) If $u(x) \neq 0$ for all $x \in X$, then $1/u$ is measurable.

iv) If $f = u + iv$, then $f : X \longrightarrow \mathbb{C}$ is measurable.

v) The previous exercises *i) ii) iii)* are also valid for $u, v : X \longrightarrow \mathbb{C}$ measurable functions and $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ continuous.

vi) If $u, v : X \longrightarrow \mathbb{R}$ and $f = u + iv$ is measurable, then u, v and $|f|$ are real measurable.

Solution: i) Let $g = \varphi \circ u$. If $V \subset \mathbb{R}$ is open, then $\varphi^{-1}(V)$ is measurable since φ is continuous, and $g^{-1}(V) = u^{-1}(\varphi^{-1}(V))$ is measurable since u is measurable.
ii) It follows from the fact that $\Phi_1(x, y) = x + y$, $\Phi_2(x, y) = xy$ are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ and $\Phi_3(x) = |x|^\alpha$ is also a continous function from $\mathbb{R}$ to $\mathbb{R}$.
iii) It follows from the fact that $\Phi : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$, given by $\Phi(x) = 1/x$, is continuous.
iv) It follows from the fact that $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{C}$, given by $\Phi(x, y) = x + iy$ is continuous.
v) The same argument used to prove i) is valid now; To prove now ii) and iii) observe that $\Phi_1(z, w) = z + w$, $\Phi_2(z, w) = zw$ are continuous from $\mathbb{C}^2$ to $\mathbb{C}$, $\Phi_3(z) = |z|^\alpha$ is continuous from $\mathbb{C}$ to $\mathbb{R}$ and $g(z) = 1/z$ is continuous from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$.
vi) It follows from the fact that $\operatorname{Re} z = (z + \bar{z})/2$, $\operatorname{Im} z = (z - \bar{z})/(2i)$ and $|z|$ are continuous functions from $\mathbb{C}$ to $\mathbb{R}$.

Problem 1.1.13 Let $(X, \mathcal{A})$ be a measurable space and let $f : X \longrightarrow \mathbb{R}$ be a function. Prove that the following assertions are equivalent:

i) $\{x \in X : f(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

ii) $\{x \in X : f(x) \geq \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

iii) $\{x \in X : f(x) < \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

iv) $\{x \in X : f(x) \leq \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

v) $f^{-1}(I) \in \mathcal{A}$ for every interval I.

vi) f is measurable, that is to say that $f^{-1}(V) \in \mathcal{A}$ for every open set V.

vii) $f^{-1}(F) \in \mathcal{A}$ for every closed set F.

viii) $f^{-1}(B) \in \mathcal{A}$ for every Borel set B.

Hint: $\mathcal{A}' = \{E \subseteq \mathbb{R} : \ f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra in $\mathbb{R}$ (in fact it is the image σ-algebra of $\mathcal{A}$) and $\mathcal{B}(\mathbb{R}) = \sigma(\{(\alpha, \infty) : \alpha \in \mathbb{R}\})$.
Solution: viii) $\Rightarrow$ i) is obvious; i) $\Rightarrow$ viii): i) means that $f^{-1}((\alpha, \infty)) \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$. Hence $\{(\alpha, \infty) : \alpha \in \mathbb{R}\} \subset \mathcal{A}'$ and so, as $\mathcal{B}(\mathbb{R})$ is the least σ-algebra containing the intervals (α, ∞), it follows that $\mathcal{B}(\mathbb{R}) \subset \mathcal{A}'$. But this fact implies viii).
Therefore we have proved that i) $\iff$ viii). The other equivalences are similar.

Problem 1.1.14 Prove that the previous problem is also valid if $f : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$. Recall that by interval, open set, closed set or Borel set in $\overline{\mathbb{R}}$ we understand the corresponding concept in $\mathbb{R}$ joining it $-\infty$, $+\infty$ or both or neither.

Solution: In this case the equivalence i) $\iff$ viii) follows from the (similar) facts that $\mathcal{A}' = \{E \subseteq \overline{\mathbb{R}} : \ f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra in $\overline{\mathbb{R}}$ and $\mathcal{B}(\overline{\mathbb{R}}) = \sigma(\{(\alpha, \infty] : \alpha \in \mathbb{R}\})$.

Problem 1.1.15 Prove that if f is a real function on a measurable space X such that $\{x \in X : \ f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Hint: Given any $\alpha \in \mathbb{R}$ there exists a sequence $\{r_n\}$ of rational numbers such that $r_n \nearrow \alpha$ as $n \to \infty$. Use problem 1.1.13.
Solution: Let $\alpha \in \mathbb{R}$ and $\{r_n\}$ be a sequence of rational numbers such that $r_n \nearrow \alpha$ as $n \to \infty$. Then $(\alpha, \infty) = \cup_n (\alpha, r_n]$ and so $f^{-1}((\alpha, \infty)) = \cup_n f^{-1}((\alpha, r_n]) \in \mathcal{A}$. Hence part i) of problem 1.1.13 is satisfied and so f is measurable.

Problem 1.1.16 Let $\mathcal{M}$ be the σ-algebra in $\mathbb{R}$ given by $\mathcal{M} = \{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$. Let g be the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 0], \\ 1 & \text{if} \quad x \in (0, 1], \\ 2 & \text{if} \quad x \in (1, \infty). \end{cases}$$

Is g measurable? How are the measurable functions $f : (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

Solution: g is not measurable since $g^{-1}(\{1\}) = (0, 1] \notin \mathcal{M}$ and $\{1\}$ is closed in $\mathbb{R}$;
If f takes only a value (f is constant) it is easy to check that f is measurable. If f is biconstant, that is to say if $f(x) = p_1$ for $x \in (-\infty, 0]$ and $f(x) = p_2$ for $x \in (0, \infty)$ then it is also easy to check that f is measurable. Finally, it is also easy to check that in any other case, f is not measurable.

Problem 1.1.17

a) Prove that if $f : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \longrightarrow \mathbb{R}$ is a continuous function, then f is measurable.

b) Prove that if $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow \mathbb{R}$ is an increasing function, then f is measurable.

c) Let $(X, \mathcal{A})$ be a measurable space. Given $A \subset X$, let χ_A be the characteristic function of A. Prove that χ_A is measurable if and only if A is measurable.

Hints: b) What can you say about $f^{-1}(I)$ when I is an interval? c) Who are $\chi_A^{-1}(0)$ and $\chi_A^{-1}(1)$?
Solution: a) As f is continuous we have that $f^{-1}(V)$ is open for all V open in $\mathbb{R}$. But open sets are borelians. Hence f is measurable.
b) As f is increasing we have that $f^{-1}(I)$ is an interval for all interval I. Hence, f is measurable by part v) of problem 1.1.13.
c) ($\Rightarrow$) If χ_A is measurable then $A = \chi_A^{-1}(\{1\}) \in \mathcal{A}$, that is to say A is measurable; ($\Leftarrow$) Assume that $A \in \mathcal{A}$ and let V be open in $\mathbb{R}$. Then, we have four possibilities: if $0, 1 \in V$ then $\chi_A^{-1}(V) = X \in \mathcal{A}$; if $0 \in V, 1 \notin V$ then $\chi_A^{-1}(V) = X \setminus A \in \mathcal{A}$; if $0 \notin V, 1 \in V$ then $\chi_A^{-1}(V) = A \in \mathcal{A}$; if $0, 1 \notin V$ then $\chi_A^{-1}(V) = \varnothing \in \mathcal{A}$. Therefore χ_A is measurable.

Problem 1.1.18 Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}} = [\infty, \infty]$. Prove that

a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

b) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

c) If $a_n \le b_n$ for all n, then $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$.

d) Show with an example that strict inequality can hold in part b).

Hint: d) Consider the sequences $a_n = (-1)^n$, $b_n = (-1)^{n+1}$.
Solution: a) For each $n \in \mathbb{N}$ we have that $\sup\{-a_n, -a_{n+1}, \ldots\} = -\inf\{a_n, a_{n+1}, \ldots\}$. To finish we pass to the limit when $n \to \infty$; b) For each $n \in \mathbb{N}$ we have that $\sup\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\} \le \sup\{a_n, a_{n+1}, \ldots\} + \sup\{b_n, b_{n+1}, \ldots\}$. To finish we pass to the limit when $n \to \infty$; c) For each $n \in \mathbb{N}$ we have that $\sup\{a_n, a_{n+1}, \ldots\} \le \sup\{b_n, b_{n+1}, \ldots\}$. To finish we pass to the limit when $n \to \infty$; d) Taking $a_n = (-1)^n$, $b_n = (-1)^{n+1}$ we have that $a_n + b_n = 0$ for all n. Hence $\limsup_{n\to\infty}(a_n + b_n) = 0$, but $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$.

Problem 1.1.19

a) Prove that if $f, g : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ are measurable functions, then $\max\{f, g\}$ and $\min\{f, g\}$ are also measurable functions.

b) Prove that if $f_n : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ is a sequence of measurable functions, then

$$\sup_n f_n\,, \qquad \inf_n f_n\,, \qquad \limsup_{n\to\infty} f_n\,, \qquad \liminf_{n\to\infty} f_n$$

are measurable functions.

c) Prove that the limit of every pointwise convergent sequence of measurable functions is measurable.

Hint: b) If $g = \sup_k f_k$ then $\{x : g(x) > \alpha\} = \cup_k \{x : f_k(x) > \alpha\}$.

Solution: a) It follows from the fact that $\max\{x, y\}$ and $\min\{x, y\}$ are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$; b) Let $g = \sup_k f_k$ and $\alpha \in \mathbb{R}$. For each $k \in \mathbb{N}$ we have that $\{x : f_k(x) > \alpha\}$ is measurable (see problem 1.1.13). But $\{x : g(x) > \alpha\} = \cup_k \{x : f_k(x) > \alpha\}$ and so this set is measurable. By part i) of problem 1.1.13 it follows that g is measurable. Similarly $\inf_k f_k$ is measurable. Finally, as $\limsup_{n\to\infty} f_n(x) = \inf_n \sup_{k\geq n} f_k(x)$ and $\liminf_{n\to\infty} f_n(x) = -\limsup_{n\to\infty}(-f_n(x))$, we deduce that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are also measurable; c) If f converges pointwisely then $\lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$ and so $\lim_{n\to\infty} f_n$ is measurable.

Problem 1.1.20 Suppose that $f, g : X \longrightarrow \mathbb{R}$ are measurable. Prove that the sets

$$\{x \in X : \; f(x) < g(x)\}, \qquad \{x \in X : \; f(x) = g(x)\}$$

are measurable.

Solution: As $g - f$ is measurable we have that $\{x \in X : \; f(x) < g(x)\} = (g - f)^{-1}(0, \infty]$ is measurable and $\{x \in X : \; f(x) = g(x)\} = (g - f)^{-1}(\{0\})$ is also measurable (see problem 1.1.13).

Problem 1.1.21 Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Hint: The set A of points at which $\{f_n\}$ converges to a finite limit verifies $A = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{i,j\geq m} \{x : |f_i(x) - f_j(x)| < \frac{1}{n}\}$.

Solution: For each pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ we have that $|f_i - f_j|$ is measurable and so (see problem 1.1.13) for each $n \in \mathbb{N}$ the set $\{x : |f_i(x) - f_j(x)| < \frac{1}{n}\} = \big(|f_i(x) - f_j(x)|\big)^{-1}\big([0, \frac{1}{n})\big)$ is measurable. Note that the set A of points at which $\{f_n\}$ converges to a finite limit verifies

$$A = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{i,j\geq m} \{x : |f_i(x) - f_j(x)| < \frac{1}{n}\}.$$

Hence, A is measurable.