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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.3: Integration on product spaces

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2 Integration Theory

2.3. Integration on product spaces

Problem 2.3.1 Prove that $f(x) = e^{-x^2} \in L^1(\mathbb{R})$ and calculate $I = \displaystyle\int_{\mathbb{R}} e^{-x^2}\, dx$.

Hint: $x^2 \geq x$ for $x \geq 1$. Relate I^2 with an integral in $\mathbb{R}^2$. Calculate this last integral using polar coordinates.

Solution: As f is continuous in $[0,1]$ we have that f is bounded in $[0,1]$ and so $f \in L^1[0,1]$. On the other hand, if $x \geq 1$ then $x \leq x^2$ and so $e^{-x^2} \leq e^{-x} \in L^1[1,\infty)$ hence, $f \in L^1[0,\infty)$. Since f is an even function, it belongs to $L^1(\mathbb{R})$. To compute the value of I we apply first Tonelli-Fubini's theorem:

$$I^2 = \Big(\int_{\mathbb{R}} e^{-x^2} dx\Big)\Big(\int_{\mathbb{R}} e^{-y^2} dy\Big) = \iint_{\mathbb{R}\times\mathbb{R}} e^{-(x^2+y^2)}\, dx\, dy\,,$$

and now we change to polar coordinates and use the monotone convergence theorem:

$$\begin{aligned} I^2 &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = 2\pi \int_0^{\infty} r\, e^{-r^2} dr = 2\pi \lim_{N\to\infty}\int_0^N r\, e^{-r^2} dr \\ &= 2\pi \lim_{N\to\infty}\Big[\frac{e^{-r^2}}{-2}\Big]_{r=0}^{r=N} = \lim_{N\to\infty} \pi\big(1-e^{-N^2}\big) = \pi \quad\Longrightarrow\quad I = \sqrt{\pi}\,. \end{aligned}$$

Problem 2.3.2 Let $A = [0,1]\times[0,1]$.

a) Prove that the function $f(x,y) = \frac{|x-y|}{(x+y)^3}$ is not integrable in A.

b) Find out if the function $f(x,y) = \frac{1}{\sqrt{xy}}$ is integrable in A and, in that case, calculate the integral $\iint f(x,y)\, dxdy$.

c) Calculate $\iint_A x\,[1+x+y]\, dxdy$ where $[t]$ denotes the integer part of t, discussing before the integrability of the function.

Hint: a) Use the change of variables $x = y+t$ and use Fubini's theorem.

Solutions: a) Using Tonelli-Fubini's theorem and then using the change of variables $x = y+t$ we obtain:

$$\begin{aligned} \iint_A \frac{|x-y|}{(x+y)^3}\, dx\, dy &= \int_0^1 \Big(\int_0^1 \frac{|x-y|}{(x+y)^3}\, dx\Big) dy = \int_0^1 \Big(\int_{-y}^{1-y} \frac{|t|}{(t+2y)^3}\, dt\Big) dy \\ &= \int_0^1 \Big(\int_{-y}^{0} \frac{-t}{(t+2y)^3}\, dt + \int_0^{1-y} \frac{t}{(t+2y)^3}\, dt\Big)\, dy\,. \end{aligned}$$

But, decomposing into simple fractions:

$$\int \frac{t}{(t+2y)^3}\, dt = \int \frac{(t+2y)-2y}{(t+2y)^3}\, dt = \int \Big(\frac{1}{(t+2y)^2} - \frac{2y}{(t+2y)^3}\Big) dt = -\frac{1}{t+2y} + \frac{y}{(t+2y)^2} + c\,,$$

and so, f is not integrable in A since $\int_0^1 \frac{1}{y}\, dy = \infty$ (see problem 2.1.8, part b3)):

$$\begin{aligned} \iint_A \frac{|x-y|}{(x+y)^3}\, dx\, dy &= \int_0^1 \Big(\Big[\frac{1}{t+2y} - \frac{y}{(t+2y)^2}\Big]_{t=-y}^{t=0} + \Big[-\frac{1}{t+2y} + \frac{y}{(t+2y)^2}\Big]_{t=0}^{t=1-y}\Big)\, dy \\ &= \int_0^1 \Big(\frac{1}{2y} - \frac{1}{y} - \frac{1}{4y} + \frac{1}{y} + \frac{1}{2y} - \frac{1}{1+y} + \frac{y}{(1+y)^2} - \frac{1}{4y}\Big)\, dy \\ &= \int_0^1 \Big(\frac{3}{4y} - \frac{1}{1+y} + \frac{y}{(1+y)^2}\Big)\, dy = \infty\,. \end{aligned}$$

b) Using again Tonelli-Fubini's theorem:

$$\iint_A \frac{1}{\sqrt{xy}}\,dx\,dy = \int_0^1 \Big(\int_0^1 \frac{1}{\sqrt{xy}}\,dx\Big)dy = \int_0^1 \frac{1}{\sqrt{y}}\big[2\sqrt{x}\,\big]_{x=0}^{x=1}dy$$
$$= 2\int_0^1 \frac{1}{\sqrt{y}}\,dy = 2\big[2\sqrt{y}\,\big]_{y=0}^{y=1} = 2\cdot 2 = 4 < \infty\,,$$

and so f is integrable in A.

c) $f(x,y) = x\,[1+x+y]$ is bounded and almost everywhere continuous and A is bounded, so f is Lebesgue-integrable; Now, given $k \in \mathbb{Z}$, $[1+x+y] = k$ if and only if $k \le x+y < k+1$, but if $(x,y) \in A$ then $0 \le x+y \le 2$. Hence, using Tonelli-Fubini's theorem:

$$\iint_A x\,[1+x+y]\,dx\,dy = \iint_{\substack{(x,y)\in A\\ 0\le x+y\le 1}} x\,dx\,dy + \iint_{\substack{(x,y)\in A\\ 1\le x+y\le 2}} 2x\,dx\,dy$$
$$= \int_0^1 \Big(\int_0^{1-x} x\,dy\Big)\,dx + \int_0^1 \Big(\int_{1-x}^1 2x\,dy\Big)\,dx$$
$$= \int_0^1 x(1-x)\,dx + \int_0^1 2x(1-(1-x))\,dx$$
$$= \Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]_{x=0}^{x=1} + \Big[\frac{2x^3}{3}\Big]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} + \frac{2}{3} = \frac{5}{6}\,.$$

Problem 2.3.3 Using Tonelli-Fubini's theorem to justify all steps, evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2}\cos\frac{\pi y}{2x}\,dx\,dy\,.$$

Hint: Prove first that $g(x,y) = x^{-3/2}\cos\frac{\pi y}{2x} \ge 0$ on $A = \{(x,y) : 0 \le y \le x \le 1\}$. Then apply Tonelli-Fubini's theorem.

Solution: Let $g(x,y) = x^{-3/2}\cos\frac{\pi y}{2x}$. If $(x,y) \in A$, then $\frac{\pi y}{2x} \in [0, \frac{\pi}{2}]$ and so $g(x,y) \ge 0$. Hence, we can apply Tonelli-Fubini's theorem:

$$\int_0^1 \int_y^1 x^{-3/2}\cos\frac{\pi y}{2x}\,dx\,dy = \int_0^1 \Big(\int_0^x x^{-3/2}\cos\frac{\pi y}{2x}\,dy\Big)\,dx$$
$$= \int_0^1 x^{-3/2}\Big(\int_0^x \cos\frac{\pi y}{2x}\,dy\Big)dx = \int_0^1 x^{-3/2}\Big[\frac{2x}{\pi}\,\sin\frac{\pi y}{2x}\Big]_{y=0}^{y=x}dx$$
$$= \int_0^1 x^{-3/2}\,\frac{2x}{\pi}\,dx = \frac{2}{\pi}\int_0^1 x^{-1/2}dx = \frac{2}{\pi}\Big[\frac{x^{1/2}}{1/2}\Big]_{x=0}^{x=1} = \frac{4}{\pi}\,.$$

Problem 2.3.4 Let us consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, with μ the counting measure.

a) Prove that $\mu \otimes \mu$ is the counting measure on $(\mathbb{N}\times\mathbb{N}, \mathcal{P}(\mathbb{N}\times\mathbb{N}))$.

b) Let us define the function

$$f(m,n) = \begin{cases} 1 & \text{if} \quad m = n\,, \\ -1 & \text{if} \quad m = n+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Check that $\int_{\mathbb{N}}(\int_{\mathbb{N}} f(m,n)\,d\mu(m))\,d\mu(n)$, and $\int_{\mathbb{N}}(\int_{\mathbb{N}} f(m,n)\,d\nu(n))\,d\mu(m)$ exist and are distinct and that $\int_{\mathbb{N}\times\mathbb{N}} |f(m,n)|\,d(\mu\otimes\mu)(m,n) = \infty$. What is the relevance of this result?

c) Do the same for the function

$$g(m,n) = \begin{cases} 1+2^{-m} & \text{if } m=n\,, \\ -1-2^{-m} & \text{if } m=n+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: a) It is clear that $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N} \times \mathbb{N})$ and that

$$(\mu \otimes \mu)(\{(m,n)\}) = (\mu \otimes \mu)(\{m\} \times \{n\}) = \mu(\{m\})\,\mu(\{n\}) = 1 \cdot 1 = 1\,.$$

Hence, $\mu \otimes \mu$ is the counting measure in $\mathbb{N} \times \mathbb{N}$.
b) Now

$$\iint_{\mathbb{N}\times\mathbb{N}} |f|\, d(\mu \otimes \mu) = \sum_{n=1}^{\infty} \sum_{m=n}^{m=n+1} 1 = \infty \qquad \Longrightarrow \qquad f \notin L^1(\mu \otimes \mu)\,.$$

Also, for fixed n,

$$\int_{\mathbb{N}} f(m,n)\, d\mu(m) = \sum_{m=1}^{\infty} f(m,n) = f(n,n) + f(n+1,n) = 1 + (-1) = 0\,,$$

and, for fixed m,

$$\int_{\mathbb{N}} f(m,n)\, d\mu(n) = \sum_{n=1}^{\infty} f(m,n) = \begin{cases} f(1,1)\,, & \text{if } m=1\,, \\ f(m,m-1)+f(m,m)\,, & \text{if } m \geq 2\,, \end{cases} = \begin{cases} 1\,, & \text{if } m=1\,, \\ 0\,, & \text{if } m \geq 2\,. \end{cases}$$

Hence,

$$\int_{\mathbb{N}} \Big(\int_{\mathbb{N}} f(m,n)\, d\mu(m) \Big)\, d\mu(n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} 0 = 0$$

and

$$\int_{\mathbb{N}} \Big(\int_{\mathbb{N}} f(m,n)\, d\mu(n) \Big)\, d\mu(m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = 1 + 0 + \cdots + 0 + \cdots = 1\,.$$

Therefore, the iterated integrals do not coincide and so Fubini's theorem can not be applied. The reason is that $f \notin L^1(\mu \otimes \mu)$. This shows that the condition of integrability in Fubini's theorem is necessary.
c) For fixed n we have:

$$\int_{\mathbb{N}} f(m,n)\, d\mu(m) = \sum_{m=1}^{\infty} f(m,n) = f(n,n) + f(n+1,n) = 1 + 2^{-n} - 1 - 2^{-n-1} = 2^{-n-1}\,,$$

and, for fixed m,

$$\begin{aligned} \int_{\mathbb{N}} f(m,n)\, d\mu(n) &= \sum_{n=1}^{\infty} f(m,n) = \begin{cases} f(1,1)\,, & \text{if } m=1\,, \\ f(m,m-1)+f(m,m)\,, & \text{if } m \geq 2\,, \end{cases} \\ &= \begin{cases} 1+2^{-1}\,, & \text{if } m=1\,, \\ -1-2^{-m}+1+2^{-m}\,, & \text{if } m \geq 2\,. \end{cases} = \begin{cases} 3/2\,, & \text{if } m=1\,, \\ 0\,, & \text{if } m \geq 2\,. \end{cases} \end{aligned}$$

Hence,

$$\int_{\mathbb{N}}\Big(\int_{\mathbb{N}} f(m,n)\,d\mu(m)\Big)\,d\mu(n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

and

$$\int_{\mathbb{N}}\Big(\int_{\mathbb{N}} f(m,n)\,d\mu(n)\Big)\,d\mu(m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \frac{3}{2} + 0 + \cdots + 0 + \cdots = \frac{3}{2}\,.$$

Hence, the iterated integrals do not coincide also in this case. Therefore, Fubini's theorem can not be applied and since $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is σ-finite the only possibility is that $f \notin L^1(\mu\otimes\mu)$ (as it can be easily verified).

Problem 2.3.5 Let $(X, \mathcal{A})$ be a measurable space an let $f : X \longrightarrow [0,\infty]$ be a positive $\mathcal{A}$-measurable function. Let

$$A_f = \{(x,y) \in X \times \mathbb{R} :\ 0 \le y \le f(x)\}.$$

a) Prove that $A_f \in \mathcal{A}\otimes\mathcal{B}(\mathbb{R})$.

b) Given a σ-finite measure μ in $(X, \mathcal{A})$ prove that $\int_X f\,d\mu$ coincides with the product measure $\pi = \mu\otimes m$ of the set A_f, where m denotes Lebesgue measure in $\mathbb{R}$.

Hints: a) Prove it first for simple functions $s(x)$ in X and later for positive functions in X. b) Use the monotone convergence theorem.

Solution: a) If $f = \chi_E$ is a characteristic function then $A_f = \big(E\times[0,1]\big)\cup\big((X\setminus E)\times\{0\}\big) \in \mathcal{A}\otimes\mathcal{B}(\mathbb{R})$. Similarly, if $s = \sum_{j=1}^m c_j\chi_{E_j}$ is a positive simple function ($c_j \ge 0$) with $E_j \in \mathcal{A}$ pairwise disjoint sets, we have that

$$A_s = \bigcup_{j=1}^{m}\big(E_j\times[0,c_j]\big)\cup\big((X\setminus E_j)\times\{0\}\big) \in \mathcal{A}\otimes\mathcal{B}(\mathbb{R})\,.$$

Finally, if $f \ge 0$ then, let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence of simple functions such that

$$0 \le s_1 \le s_2 \cdots \le s_n \le \cdots \nearrow f\,, \qquad \text{as } n\to\infty\,.$$

Then

$$A_f = \{(x,y)\in X\times\mathbb{R} :\ 0\le y\le f(x)\} = \bigcup_{j=1}^{\infty}\{(x,y)\in X\times\mathbb{R} :\ 0\le y\le s_j(x)\} = \bigcup_{j=1}^{\infty} A_{s_j} \in \mathcal{A}\otimes\mathcal{B}(\mathbb{R})\,.$$

b) If $s = \sum_{j=1}^m c_j\chi_{E_j}$ is a positive simple function ($c_j \ge 0$) with $E_j \in \mathcal{A}$ pairwise disjoint sets, we have that

$$(\mu\otimes m)(A_s) = \sum_{j=1}^{m}(\mu\otimes m)(E_j\times[0,c_j]) + (\mu\otimes m)(X\setminus E_j)\times\{0\}) = \sum_{j=1}^{m} c_j\mu(E_j) + 0 = \int_X s\,d\mu\,, \quad (1)$$

since $m(\{0\}) = 0$.
If $f \ge 0$, let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence of positive measurable functions with $s_n \nearrow f$ as $n\to\infty$. Then, since A_f is equal to the increasing union of the sets A_{s_j}, and using (1) and the monotone convergence theorem:

$$(\mu\otimes m)(A_f) = \lim_{n\to\infty}(\mu\otimes m)(A_{s_n}) = \lim_{n\to\infty}\int_X s_n\,d\mu = \int_X f\,d\mu\,.$$

Problem 2.3.6 Let $X = Y = [0,1]$, $\mathcal{A}_1\,,\ \mathcal{A}_2 = \mathcal{B}([0,1])$, μ the Lebesgue measure on $\mathcal{A}_1$, ν the counting measure on $\mathcal{A}_2$. In the measure space $(X \times Y, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu \otimes \nu)$ we consider the set $V = \{(x,y) : x = y\}$. Check that $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$. However

$$\int_Y d\nu \int_X \chi_V \, d\mu = 0\,, \qquad \int_X d\mu \int_Y \chi_V \, d\nu = 1\,.$$

What hypothesis of Fubini's theorem does not hold?

Hint: If $V_n = (I_1 \times I_1) \cup \cdots \cup (I_n \times I_n) \cup \{(1,1)\}$ being $I_j = [\frac{j-1}{n}, \frac{j}{n})$ $j = 1, 2, \ldots, n$, then $V = \cap_1^\infty V_n\,.$
Solution: For each $n \in \mathbb{N}$ let $V_n = (I_1 \times I_1) \cup \cdots \cup (I_n \times I_n) \cup \{(1,1)\}$ where $I_j = [\frac{j-1}{n}, \frac{j}{n})$ $j = 1, 2, \ldots, n$. Then it is clear $V \subset V_n$ but it is also easy to check that $V = \cap_{n=1}^\infty V_n$. Hence, as V_n is a union of products of semiopen intervals and a point we have that $V_n \in \mathcal{A}_1 \otimes \mathcal{A}_2$ for all $n \in \mathbb{N}$ and therefore also $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$.
On the other hand,

$$\int_Y d\nu \int_X \chi_V \, d\mu = \int_Y \Big(\int_0^1 \chi_V(x,y)\, dx \Big) d\nu(y) = \int_Y m(\{y\})\, d\nu(y) = \int_Y 0 \; d\nu(y) = 0\,,$$

and

$$\int_X d\mu \int_Y \chi_V \, d\nu = \int_0^1 \Big(\int_Y \chi_V(x,y)\, d\nu(y) \Big)\, dx = \int_0^1 \nu(\{x\})\, dx = \int_0^1 1 \; dx = 1\,.$$

Therefore the iterated integrals do not coincide and so we can not apply Fubini's theorem. Since $\chi_V \geq 0$ the only possible reason is that $(Y, \mathcal{B}([0,1]), \nu)$ is not σ-finite and this example shows that the σ-finiteness hypothesis is necessary in Tonelli-Fubini's theorem.

Problem 2.3.7 Let $(X_k, \mathcal{A}_k, \mu_k)$ be σ-finite measure spaces, $k = 1, 2\ldots, n$. Let $f_k : X_k \longrightarrow [0, \infty]$ be positive $\mathcal{A}_k$-measurable functions, $k = 1, 2\ldots, n$.

a) Prove that the product function $h = f_1 f_2 \ldots f_n : X_1 \times \cdots \times X_n \longrightarrow [0,\infty]$ given by

$$h(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

is $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$-measurable and that

$$\int_{X_1 \times \cdots \times X_n} (f_1 f_2 \ldots f_n)\, d\mu_1 \otimes \cdots \otimes d\mu_n = \prod_{i=1}^n \int_{X_i} f_i \, d\mu_i\,. \tag{2}$$

b) Use this formula to compute the integral $\displaystyle\int_{\mathbb{R}^n} e^{-\|x\|^2}\, dx\,.$

c) Calculate again this integral using the formula for radial functions in Problem 2.2.26 and from this obtain the value of $\Omega_n = m(B_n)$, the n-dimensional Lebesgue measure of the unit ball B_n of $\mathbb{R}^n$.

d) Prove that part a) also holds when the functions $f_1, \ldots, f_k$ are not positive but $f_k \in L^1(\mu_k)$, $k = 1, 2\ldots, n$.

Hints: a) Consider the functions $F_i(x_1, x_2, \ldots, x_n) := f_i(x_i)$ and use Fubini's theorem for positive functions. b) Use a) and problem 2.3.1. c) Use Euler's Gamma function and that $x\Gamma(x) = \Gamma(x+1)$. d) Use Fubini's theorem.

Solution: a) For $k = 1, \ldots, n$, let $F_i : X_1 \times X_2 \times \cdots X_n \longrightarrow [0, \infty]$ given by $F_k(x_1, x_2, \ldots, x_n) := f_k(x_k)$. Then, if $V \subseteq [0, \infty]$ is open, then

$$F_k^{-1}(V) = X_1 \times \cdots \times X_{k-1} \times f_k^{-1}(V) \times X_{k+1} \times \cdots \times X_n \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \,,$$

since $f_k^{-1}(V) \in \mathcal{A}_k$ because f_k is $\mathcal{A}_k$-measurable by hypothesis. Finally, as $h = F_1 F_2 \cdots F_n$, we obtain that h is $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$-measurable because h is a product of measurable functions. Finally, (2) follows from Tonelli-Fubini's theorem.
b) Using (2) and problem 2.3.1 we have

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\|x\|^2}\, dx &= \int_{\mathbb{R}\times\cdots\times\mathbb{R}} e^{-x_1^2} e^{-x_2^2} \cdots e^{-x_n^2}\, dx_1 \ldots dx_n \\ &= \prod_{k=1}^{n} \int_{\mathbb{R}} e^{-x_k^2}\, dx_k = \Big(\int_R e^{-x^2}\, dx \Big)^n = (\sqrt{\pi})^n = \pi^{n/2}.\end{aligned}$$

c) Using the formula in problem 2.2.26 we have

$$\int_{\mathbb{R}^n} e^{-\|x\|^2}\, dx = n\,\Omega_n \int_0^\infty e^{-r^2} r^{n-1}\, dr = \frac{1}{2}\, n\,\Omega_n \int_0^\infty u^{n/2-1} e^{-u}\, du \,,$$

where we have done the change of variable $u = r^2$. Using now the Euler Gamma-function $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$, we obtain that:

$$\int_{\mathbb{R}^n} e^{-\|x\|^2}\, dx = \frac{1}{2}\, n\,\Omega_n\, \Gamma\Big(\frac{n}{2}\Big) .$$

From this and the formula obtained in b) we deduce that

$$\frac{1}{2}\, n\,\Omega_n\, \Gamma\Big(\frac{n}{2}\Big) = \pi^{n/2} \qquad \Longrightarrow \qquad \Omega_n = \frac{2\pi^{n/2}}{n\Gamma\big(\frac{n}{2}\big)} = \frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)} .$$

where we have used the well-known formula $\Gamma(x+1) = x\Gamma(x)$.
d) As $|h| = |f_1| \cdots |f_n|$ and since $f_k \in L^1(\mu_k)$, $k = 1, 2 \ldots, n.$, we obtain from a) that $h \in L^1(\mu_1 \otimes \cdots \otimes \mu_n)$:

$$\int_{X_1\times\cdots\times X_n} |h|\, d\mu_1 \otimes \cdots \otimes d\mu_n = \int_{X_1\times\cdots\times X_n} |f_1|\,|f_2| \cdots |f_n|\, d\mu_1 \otimes \cdots \otimes d\mu_n = \prod_{i=1}^{n} \int_{X_i} |f_i|\, d\mu_i < \infty \,.$$

Hence, we can apply now Fubini's theorem to obtain (2) for general functions $f_1, \ldots, f_n$.

Problem 2.3.8 Let us consider the Lebesgue measure on $\mathbb{R}^2$. Let $A = [a, b] \times [c, d]$ and let f be continuous on A. Prove that

$$\int_A f\, dm = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx \,.$$

Solution: Since f is continuous in the compact set A, we have that f is bounded in A and so, as A has finite Lebesgue measure, f is Lebesgue-integrable in A. The formula is now a direct consequence of Fubini's theorem.

Problem 2.3.9 Let

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

Check that

$$\int_0^1 dx \int_0^1 f(x,y)\,dy = \frac{\pi}{4}\,, \qquad \int_0^1 dy \int_0^1 f(x,y)\,dx = -\frac{\pi}{4}\,.$$

What hypothesis of Fubini's theorem does not hold?

Solution: The iterated integrals are

$$\int_0^1 dx \int_0^1 f(x,y)\,dy = \int_0^1 dx \int_0^1 \frac{\partial}{\partial y}\Big(\frac{y}{x^2+y^2}\Big)\,dy = \int_0^1 \int_0^1 \Big[\frac{y}{x^2+y^2}\Big]_{y=0}^{y=1} dx = \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

and

$$\int_0^1 dx \int_0^1 f(x,y)\,dy = \int_0^1 dy \int_0^1 \frac{\partial}{\partial x}\Big(\frac{-x}{x^2+y^2}\Big)\,dx = \int_0^1 \int_0^1 \Big[\frac{-x}{x^2+y^2}\Big]_{x=0}^{x=1} dy = \int_0^1 \frac{dx}{1+x^2} = -\frac{\pi}{4}\,.$$

Hence, Fubini's theorem can not be applied. The reason is, a fortiori, that $f \notin L^1([0,1]\times[0,1])$.

Problem 2.3.10 Let us define the function $f : [-1,1]\times[-1,1] \longrightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2}\,, & (x,y) \neq (0,0)\,, \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

Check that

$$\int_{-1}^1 dx \int_{-1}^1 f(x,y)\,dy = \int_{-1}^1 dy \int_{-1}^1 f(x,y)\,dx\,,$$

but however f is not integrable in $[-1,1]\times[-1,1]$. Why is relevant this exercise?

Solution: Since $f(x,y)$ is odd in both variables and the domain is symmetric with respect to both variables we have that both iterated integrals vanish. However, $f \notin L^1([-1,1]\times[-1,1])$:

$$\int_{-1}^1 \int_{-1}^1 |f(x,y)|\,dx\,dy \geq \int_0^1 \int_0^{2\pi} \frac{|r^2 \sin\theta\cos\theta|}{r^4}\, r\,dr\,d\theta = 8\pi\Big(\int_0^{2\pi} \sin\theta\cos\theta d\theta\Big)\Big(\int_0^1 \frac{dr}{r}\Big) = \infty.$$

This fact shows that Fubini's theorem is not an equivalence.

Problem 2.3.11 Sometimes, Fubini's Theorem can be used as a tool to show that a one variable integral converges to a certain value, by *transforming* the simple integral into a double one and, in a justified way, exchange order of integration. With this idea in mind and using that

$$\frac{1}{x} = \int_0^\infty e^{-tx} dt,$$

show that

$$\lim_{R\to\infty} \int_0^R \frac{\sin x}{x}\,dx = \frac{\pi}{2}\,.$$

Hint: Consider the function $f(x,t) = e^{-xt}\sin x$ defined in the set $(0,R)\times(0,\infty)$ and prove that

$$\int_0^R dx \int_0^\infty f(x,t)\,dt = \int_0^R \frac{\sin x}{x}\,dx < \infty \quad \text{but} \quad \int_0^\infty dt \int_0^R f(x,t)\,dx = \frac{\pi}{2} - \int_0^\infty \frac{e^{-Rt}(\cos R + t\sin R)}{1+t^2}\,dt.$$

Finally, using dominated convergence, prove that this last integral converges to zero as $R \to \infty$.
Solution: Consider the function $f(x,t) = e^{-xt}\sin x$ defined in the set $(0,R)\times(0,\infty)$. The iterated integrals are

$$\int_0^R dx \int_0^\infty f(x,t)\,dt = \int_0^R \sin x\Big(\int_0^\infty e^{-xt}dt\Big)\,dx = \int_0^R \sin x\,\Big[-\frac{e^{-xt}}{x}\Big]_{t=0}^{t=\infty} dx = \int_0^R \frac{\sin x}{x}\,dx$$

and, integrating by parts, and using the monotone convergence theorem:

$$\begin{aligned}\int_0^\infty dt \int_0^R f(x,t)\,dx &= \int_0^\infty \frac{1-e^{-Rt}\cos R - te^{-Rt}\sin R}{1+t^2}\,dt\\ &= \lim_{N\to\infty}\big[\arctan t\big]_{t=0}^{t=N} - \int_0^\infty \frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\,dt\\ &= \frac{\pi}{2} - \int_0^\infty \frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\,dt\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^R\int_0^\infty |f(x,t)|\,dx\,dt &\le \int_0^R |\sin x|\,\Big(\int_0^\infty e^{-xt}dt\Big)\,dx\\ &= \int_0^R |\sin x|\,\Big[-\frac{e^{-xt}}{x}\Big]_{t=0}^{t=\infty} dx = \int_0^R \frac{|\sin x|}{x}\,dx < \infty\end{aligned}$$

since $|\sin x|/x$ is continuous in $[0,R]$ and so is integrable. By Fubini's theorem, both iterated integrals are equal:

$$\int_0^R \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \int_0^\infty \frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\,dt\,.$$

But

$$\Big|\frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\Big| \le \frac{e^{-t}(1+t)}{1+t^2} \in L^1(0,\infty)$$

and by the dominated convergence theorem we conclude that

$$\begin{aligned}\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\,dx &= \frac{\pi}{2} - \lim_{R\to\infty}\int_0^\infty \frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\,dt\\ &= \frac{\pi}{2} - \int_0^\infty \Big(\lim_{R\to\infty}\frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\Big)\,dt = \frac{\pi}{2} - 0 = \frac{\pi}{2}\,.\end{aligned}$$

Problem 2.3.12

a) Prove that the function $f(x,y) = e^{-y}\sin 2xy$ is integrable in $A = [0,1]\times[0,\infty)$.

b) Prove that

$$\int_0^1 e^{-y}\sin 2xy\,dx = \frac{e^{-y}}{y}\,\sin^2 y\,,\qquad \int_0^\infty e^{-y}\sin 2xy\,dy = \frac{2x}{1+4x^2}\,.$$

c) Using Fubini's theorem, prove that:

$$\int_0^\infty e^{-y}\,\frac{\sin^2 y}{y}\,dy = \frac{1}{4}\,\log 5\,.$$

Solution: a) Using Fubini's theorem for positive functions we have that

$$\int_0^1 \int_0^\infty |f(x,y)|\,dx\,dy \le \int_0^1 \int_0^\infty e^{-y}\,dy\,dx = \int_0^1 dx \int_0^\infty e^{-y}\,dy < \infty\,.$$

b) First,

$$\int_0^1 e^{-y}\sin 2xy\,dx = e^{-y}\Big[-\frac{\cos(2xy)}{2y}\Big]_{x=0}^{x=1} = e^{-y}\,\frac{1-\cos 2y}{2y} = \frac{e^{-y}}{y}\,\sin^2 y\,.$$

Secondly, integrating by parts with $u = \sin(2xy) \implies du = 2x\cos(2xy)\,dy$, $dv = e^{-y}dy \implies v = -e^{-y}$, we have

$$\int_0^\infty e^{-y}\sin 2xy\,dy = \lim_{N\to\infty}[-e^{-y}\sin(2xy)]_{y=0}^{y=N} + \int_0^\infty 2xe^{-y}\cos(2xy)\,dy = 2x\int_0^\infty e^{-y}\cos(2xy)\,dy\,.$$

Using parts again: $u = \cos(2xy) \implies du = -2x\sin(2xy)\,dy$, $dv = e^{-y}dy \implies v = -e^{-y}$, we have

$$\begin{aligned}\int_0^\infty e^{-y}\sin 2xy\,dy &= 2x\int_0^\infty e^{-y}\cos(2xy)\,dy = 2x\lim_{N\to\infty}\Big([-e^{-y}\cos(2xy)]_{y=0}^{y=N} - \int_0^\infty 2xe^{-y}\sin(2xy)\,dy\Big)\\ &= 2x - 4x^2\int_0^\infty e^{-y}\sin 2xy\,dy\end{aligned}$$

and so

$$\int_0^\infty e^{-y}\sin 2xy\,dy = \frac{2x}{1+4x^2}\,.$$

c) Using now part b) and Fubini's theorem, we have:

$$\begin{aligned}\int_0^\infty e^{-y}\,\frac{\sin^2 y}{y}\,dy &= \int_0^\infty\int_0^1 e^{-y}\sin 2xy\,dx\,dy = \int_0^1\int_0^\infty e^{-y}\sin 2xy\,dy\,dx\\ &= \int_0^1 \frac{2x}{1+4x^2}\,dx = \Big[\frac14\,\log(1+4x^2)\Big]_{x=0}^{x=1} = \frac14\,\log 5\,.\end{aligned}$$

Problem 2.3.13 Let μ be the Lebesgue measure on $[0,1]$ and ν be the counting measure on $\mathbb{N}$. Let us define $G : [0,1]\times\mathbb{N} \longrightarrow \mathbb{R}$ by $G(x,n) = \left(\frac{x}{2}\right)^n$.

a) Prove that for $0 < a \le 1$ we have that $G^{-1}((-\infty,a)) = \cup_n([0,2a^{1/n})\times\{n\})$.

b) Deduce that G is $\mu\otimes\nu$-measurable.

c) Use Fubini's theorem to prove that

$$\sum_{n=1}^\infty \frac{1}{(n+1)2^n} = 2\log 2 - 1.$$

Hint: b) Use Problem 1.1.13

Solution: a) Let $0 < a \le 1$. Then

$$(x,n)\in G^{-1}((-\infty,a)) \iff G(x,n) = \left(\frac{x}{2}\right)^n < a \iff x < 2a^{1/n} \iff (x,n)\in[0,2a^{1/n})\times\{n\}\,.$$

b) If $0 < a \leq 1$, then $G^{-1}((-\infty, a)) = \cup_{n=1}^{\infty}\big([0, 2a^{1/n}) \times \{n\}\big)$ is $\mu \otimes \nu$-measurable. And, also if $a > 1$, then $G^{-1}((-\infty, a)) = [0,1] \times \mathbb{N}$ is $\mu \otimes \nu$-measurable. Note that $G^{-1}((-\infty, a)) = \varnothing$ if $a \leq 0$.
c) As G is positive and the spaces are σ-finite, we can apply Tonelli-Fubini's theorem. The iterated integrals are:

$$\iint_{[0,1]\times\mathbb{N}} G\, d\mu \otimes d\nu = \int_{\mathbb{N}} \Big(\int_0^1 \Big(\frac{x}{2}\Big)^n dx \Big)\, d\nu(n) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Big[\frac{x^{n+1}}{n+1}\Big]_{x=0}^{x=1} = \sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$$

and

$$\begin{aligned}\iint_{[0,1]\times\mathbb{N}} G\, d\mu \otimes d\nu &= \int_0^1 \Big(\int_{\mathbb{N}} \Big(\frac{x}{2}\Big)^n d\nu(n) \Big)\, dx = \int_0^1 \sum_{n=1}^{\infty} \Big(\frac{x}{2}\Big)^n dx = \int_0^1 \frac{x/2}{1-x/2}\, dx \\ &= \int_0^1 \Big(-1 + \frac{2}{2-x} \Big)\, dx = \big[-x - 2\log(2-x) \big]_{x=0}^{x=1} = 2\log 2 - 1\,.\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)2^n} = 2\log 2 - 1.$$

Problem 2.3.14 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1\,, & \text{if } x \in [0,1] \cap \mathbb{Q},\ y \in [0,1]\,, \\ 0\,, & \text{if } x \in [0,1] \setminus \mathbb{Q},\ y \in [0,1]\,. \end{cases}$$

a) Prove that f is measurable with respect to Lebesgue σ-algebra.

b) Prove that $\displaystyle\iint_{[0,1]^2} f(x,y)\, dx\, dy = 0$.

Solution: a) Let us observe that $F = \chi_{(\mathbb{Q}\cap[0,1])\times[0,1]}$. Hence, as $Q \cap [0,1]$ is Lebesgue measurable, then f also is (see problem 1.1.17).
b) Since $f \geq 0$, by Tonelli-Fubini's theorem:

$$\iint_{[0,1]^2} f(x,y)\, dxdy = m(Q \cap [0,1])\, m([0,1]) = 0$$

since $Q \cap [0,1]$ is countable.

Problem 2.3.15 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1\,, & \text{if } xy \in \mathbb{Q}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

a) Prove that f is measurable with respect to Lebesgue σ-algebra.

b) Prove that $\displaystyle\iint_{[0,1]^2} f(x,y)\, dxdy = 0$.

Solution: a) Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$ and $E_k := \{(x,y) \in [0,1] \times [0,1] : xy = r_k\}$. Then $f = \chi_E$, where $E = \cup_{k=1}^{\infty} E_k$. Then, as $g(x,y) = xy$ is continuous, then $E_k = g^{-1}(\{r_k\})$ is closed and so, E_k is Lebesgue measurable. Hence, $E = \cup_{k=1}^{\infty} E_k$ is also Lebesgue measurable and so, $f = \chi_E$ is Lebesgue measurable.
b) We have that

$$m(E_k) = \int_0^1 \Big(\int_0^1 \chi_{E_k}\, dy \Big)\, dx = \int_0^1 m(\{y : xy = r_k\})\, dx = \int_0^1 m\Big(\Big\{\frac{r_k}{x}\Big\}\Big)\, dx = \int_0^1 0\, dx = 0$$

and so

$$\iint_{[0,1]^2} f(x,y)\, dx\, dy = m(E) = \sum_{k=1}^{\infty} m(E_k) = 0\,.$$

Problem 2.3.16 Let us consider the measure space $([0,1] \times [0,1], \mathcal{M}, m_2)$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and m_2 is the two-dimensional Lebesgue measure. Given $E \in \mathcal{M}$, let us denote

$$E_x = \{y \in [0,1] :\ (x,y) \in E\}\,, \qquad E^y = \{x \in [0,1] :\ (x,y) \in E\}\,.$$

Let m_1 denote Lebesgue measure on $[0,1]$. Prove that if $E \in \mathcal{M}$ verifies that $m_1(E_x) \le 1/2$ for almost all $x \in [0,1]$, then

$$m_1(\{y \in [0,1] :\ m_1(E^y) = 1\}) \le \frac{1}{2}\,.$$

Hint: Apply Fubini's theorem to the function $f = \chi_E$ and consider the set $A = \{y \in [0,1] : m_1(E^y) = 1\}$.
Solution: Let $f = \chi_E$ and $A = \{y \in [0,1] : m_1(E^y) = 1\}$. Then

$$m_2(E) = \int_0^1 \Big(\int_0^1 \chi_E\, dy \Big)\, dx = \int_0^1 m_1(E_x)\, dx \le \frac{1}{2}$$

and, by Tonelli-Fubini's theorem, also:

$$m_2(E) = \int_0^1 \Big(\int_0^1 \chi_E\, dx \Big)\, dy = \int_0^1 m_1(E^y)\, dy \ge \int_A m_1(E^y)\, dy = \int_A dy = m_1(A)\,.$$

Hence, $m_1(A) \le 1/2$.

Problem 2.3.17 Let $f \in L^1(0,\infty)$. Given $\alpha > 0$, let us define $g_\alpha(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$ for $x > 0$. Check that $\alpha \int_0^y g_\alpha(x) dx = g_{\alpha+1}(y)$ for $y > 0$.

Hint: Check that you can apply Tonelli-Fubini's theorem.
Solution: If $f(t) \ge 0$, then Tonelli-Fubini's theorem gives that the formula holds:

$$\begin{aligned}
\alpha \int_0^y g_\alpha(x)\, dx &= \alpha \int_0^\infty \chi_{[0,y]}(x) \Big(\int_0^\infty (x-t)^{\alpha-1} \chi_{[0,x]}(t)\, f(t)\, dt \Big)\, dx \\
&= \alpha \int_0^\infty \chi_{[0,y]}(t) \Big(\int_0^\infty (x-t)^{\alpha-1} \chi_{[t,y]}(x)\, f(t)\, dx \Big)\, dt \qquad (3) \\
&= \alpha \int_0^y \Big[\frac{(x-t)^\alpha}{\alpha} \Big]_{x=t}^{x=y} f(t)\, dt = \int_0^y (y-t)^\alpha f(t)\, dt = g_{\alpha+1}(y)\,.
\end{aligned}$$

For general $f \in L^1(0,\infty)$ we have that (3) holds for $|f| \geq 0$ and so,

$$\int_0^\infty \int_0^\infty |(x-t)^{\alpha-1} f(t)|\, \chi_{\{(x,t):0\leq t\leq x\leq y\}}\, dx\, dt = \int_0^\infty \chi_{[0,y]}(x) \Big(\int_0^\infty (x-t)^{\alpha-1} \chi_{[0,x]}(t)\, |f(t)|\, dt \Big)\, dx$$
$$= \frac{1}{\alpha} \int_0^y (y-t)^\alpha |f(t)|\, dt \leq \frac{y^\alpha}{\alpha} \int_0^\infty |f(t)|\, dt < \infty\,.$$

Hence, the function $(x-t)^{\alpha-1} f(t) \in L^1(\{(x,t) : 0 \leq t \leq x \leq y\})$ for each $y > 0$ and therefore we can use Tonelli-Fubini's theorem for general f in the computations in (3).

Problem 2.3.18 Let f and g be Lebesgue integrable functions on $[0,1]$, and let F and G be the integrals

$$F(x) = \int_0^x f(t)\, dt\,, \qquad G(x) = \int_0^x g(t)\, dt\,.$$

Use Fubini's theorem to prove that

$$\int_0^1 F(x) g(x)\, dx = F(1)G(1) - \int_0^1 f(x)G(x)\, dx\,.$$

Solution: As a direct consequence of problem 2.3.7 we have that $f(t)\, g(x) \in L^1([0,1] \times [0,1])$ and applying Fubini's theorem we get that:

$$\begin{aligned}
\int_0^1 F(x)\, g(x)\, dx &= \int_0^1 g(x) \Big(\int_0^x f(t)\, dt \Big)\, dx = \int_0^1 f(t) \Big(\int_t^1 g(x)\, dx \Big)\, dt \\
&= \int_0^1 f(t) \Big(\int_0^1 g(x)\, dx - \int_0^t g(x)\, dx \Big)\, dt \\
&= \int_0^1 f(t) \Big(\int_0^1 g(x)\, dx \Big)\, dt - \int_0^1 f(t) \Big(\int_0^t g(x)\, dx \Big)\, dt \\
&= \Big(\int_0^1 g(x)\, dx \Big) \Big(\int_0^1 f(t)\, dt \Big) - \int_0^1 f(t)\, G(t)\, dt = F(1)\, G(1) - \int_0^1 f(t)\, G(t)\, dt\,.
\end{aligned}$$

Problem 2.3.19* Apply Fubini's theorem to obtain the following recurrence formula for n-dimensional measure Ω_n of the unit ball B_n of $\mathbb{R}^n$:

$$\Omega_n = \sqrt{\pi}\, \Omega_{n-1} \frac{\Gamma\big(\frac{n+1}{2}\big)}{\Gamma\big(\frac{n}{2}+1\big)}\,.$$

Hint: $\Omega_n = \int_{-1}^1 m_{n-1}(B_{x_1})\, dx_1$ where $B_{x_1} = \{\bar{x} \in \mathbb{R}^{n-1} : \|\bar{x}\| < (1-x_1^2)^{1/2}\}$. Relate $m_{n-1}(B_{x_1})$ with Ω_{n-1} and use the Euler's β-function $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$ and the formula $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx$ is the Euler Γ-function.

Problem 2.3.20* Given $x \in \mathbb{R}^n \setminus \{0\}$, let us consider its polar coordinates (r, x') where $r = \|x\| \in (0,\infty)$, $x' = x/\|x\| \in S_{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The mapping

$$\varphi : \mathbb{R}^n \setminus \{0\} \longrightarrow (0,\infty) \times S_{n-1} \qquad \text{given by } \varphi(x) = (r, x')$$

is a bijection. Prove that

a) If μ is the image measure under φ of the Lebesgue measure on $\mathbb{R}^n \setminus \{0\}$, then

$$\mu(E \times U) = \sigma(U) \int_E r^{n-1} dr\,, \qquad \text{for all borel sets } E \subseteq (0,\infty),\ U \subseteq S_{n-1}\,.$$

b) If $f : \mathbb{R}^n \setminus \{0\} \longrightarrow [0,\infty]$ is a positive measurable function, then

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty r^{n-1} dr \int_{S_{n-1}} f(rx')\, d\sigma(x')$$

where σ is the $(n-1)$-dimensional Lebesgue measure on S_{n-1}.

c) Given $f(x) = |x_1 x_2 \cdots x_n|$, use Fubini's theorem to obtain a recurrence formula relating $I_n = \int_{B_n} f(x)\, dx$ with I_{n-1}. Deduce the value of I_n.

d) Apply parts b) and c) to evaluate $J_n = \int_{S_{n-1}} f(x')\, d\sigma(x')$, .

Hints: a) For each fixed Borel set $U \subset S_{n-1}$, as a consequence of Caratheodory-Hopf's theorem, it suffices to prove that both sides of the identity coincide for semi-intervals $E = [a,b)$. b) Observe that $f = f \circ \varphi \circ \varphi^{-1}$ and use first problem **??**, part a) and later Fubini's theorem.
Solution: c) $I_n = I_{n-1}/n$ and so $I_n = 1/n!$. d) $I_n = J_n/(2n)$ and so $J_n = 2/(n-1)!$.