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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.4: Decomposition of measures

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2 Integration Theory

2.4. Decomposition of measures

Problem 2.4.1 Let μ, λ be measures defined on the same σ-algebra. Prove that if we have at the same time $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda \equiv 0$.

Solution: As $\lambda \perp \mu$ we have that $X = A \cup B$ with $A \cap B = \varnothing$, $\lambda(A) = 0$ and $\mu(B) = 0$. But then, as $\lambda \ll \mu$ we have that $\lambda(B) = 0$. Hence, $\lambda(X) = \lambda(A) + \lambda(B) = 0 \implies \lambda \equiv 0$.

Problem 2.4.2 Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}([0, 1])$, m the Lebesgue measure on $\mathcal{M}$, μ the counting measure on $\mathcal{M}$.

a) Prove that $m \ll \mu$ but $dm \neq f\, d\mu$ for all f.

b) Prove that m has not Radon-Nikodym decomposition with respect to μ.

c) What hypothesis fails to apply in Radon-Nikodym theorem?

Hints: a) Suppose that there exists such an f. Then, there exists x_0 such that $f(x_0) \neq 0$. Consider the set $A = \{x_0\}$. b) How must be a measure which is mutually orthogonal with μ?

Solution: a) If $A \subset [0, 1]$ verifies $\mu(A) = 0$, then $A = \varnothing$ and so, $m(A) = 0$. Hence, $m \ll \mu$. Now, let $f \neq 0$. Then $\exists\, x_0 \in [0, 1]$ such that $f(x_0) \neq 0$. Let $A = \{x_0\}$, then $m(A) = 0$, but

$$\int_A f\, d\mu = \int_{\{x_0\}} f\, d\mu = f(x_0) \neq 0 \qquad \Longrightarrow \qquad m(A) \neq \int_A f\, d\mu\,.$$

c) What is failing here to apply Radon-Nikodym theorem is that μ is not σ-finite.

Problem 2.4.3 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N}$ be a σ-subalgebra of $\mathcal{M}$ and let ν be the restriction of μ to $\mathcal{N}$. If $0 \leq f \in L^1(\mu)$ prove that there exists g $\mathcal{N}$-measurable, $0 \leq g \in L^1(\nu)$, such that $\int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. Besides, g is unique module alterations in ν-null sets.

Hint: Consider the measure $\lambda(E) = \int_E f\, d\mu$ for $E \in \mathcal{N}$. The function g is called the *conditioned expected value $E(f|\mathcal{N})$ of f with respect to $\mathcal{N}$.*

Solution: Let us define $\lambda(E) = \int_E f\, d\mu$ for $E \in \mathcal{N}$. Then λ is a positive finite measure in X. As $\lambda \ll \mu$, by Radon-Nikodym theorem, there exists an $\mathcal{N}$-measurable function, $g :\longrightarrow [0, \infty]$ such that $d\lambda = g\, d\nu$ or equivalently, such that

$$\lambda(E) = \int_E g\, d\nu \quad \text{for all } E \in \mathcal{N} \qquad \Longrightarrow \qquad \int_E f\, d\mu = \int_E g\, d\nu \quad \text{for all } E \in \mathcal{N}\,.$$

Observe that, as $f \in L^1(\mu)$, we have that also $g \in L^1(\nu)$. Besides, g is unique up to zero-measure sets, since

$$\int_E g_1\, d\mu = \int_E g_2\, d\nu \quad \forall\, E \in \mathcal{N} \qquad \Longrightarrow \qquad \int_E (g_1 - g_2)\, d\nu = 0 \quad \forall\, E \in \mathcal{N} \qquad \Longrightarrow \qquad g_1 = g_2\ \nu\text{-a.e.}.$$

Problem 2.4.4 Let μ and ν be finite positive measures on the measurable space $(X, \mathcal{A})$. Show that there is a nonnegative measurable function f on X such that for all $A \in \mathcal{A}$

$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu\,.$$

Hint: The statement is equivalent to $\mu(A) = \int_A f\, d(\mu+\nu)$.
Solution: Since μ and ν are positive, we have that $\mu \ll \mu+\nu$ and so, by Radon-Nikodym theorem, there exists a positive function $f \in L^1(\mu+\nu)$ such that

$$\mu(A) = \int_A f\, d(\mu+\nu)\,, \quad \forall\, A \in \mathcal{A} \qquad \Longrightarrow \qquad \int_A (1-f)\, d\mu = \int_A f\, d\nu\,, \quad \forall\, A \in \mathcal{A}\,.$$

Problem 2.4.5 Let m be the Lebesgue measure on the real line $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$ define

$$\mu(E) = \int_E \frac{1}{1+x^2}\, dm(x)\,.$$

a) Show that $m \ll \mu$.
b) Compute the Radon-Nikodym derivative $h = dm/d\mu$.

Hints: a) Use Problem 2.1.1. b) Given a Lebesgue-measurable set E, consider the function $f(x) = \frac{1}{1+x^2}\, \chi_E(x)$ and use problem 2.2.21.

Solution: a) If $\mu(E) = 0$ then, by problem 2.1.1, we have that $m\big(\{x \in E : \frac{1}{1+x^2} \neq 0\}\big) = 0$ and so, $m(E) = 0$. Hence, $m \ll \mu$.
b) As $m \ll \mu$, by Radon-Nikodym theorem, there exists a positive function $f \in L^1(\mu)$ such that, for any Lebesgue-measurable set E,

$$m(E) = \int_E f(x)\, d\mu(x) \qquad \text{and} \qquad \int_{\mathbb{R}} g\, dm = \int_{\mathbb{R}} g\, f\, d\mu\,, \quad \forall\, g \in L^1(m)\,.$$

by problem 2.2.21. Then, for any Lebesgue-measurable set E

$$\mu(E) = \int_E \frac{1}{1+x^2}\, dm(x) = \int_{\mathbb{R}} \frac{1}{1+x^2}\, \chi_E(x)\, dm(x) = \int_{\mathbb{R}} \frac{f(x)}{1+x^2}\, \chi_E(x)\, d\mu(x) = \int_E \frac{f(x)}{1+x^2}\, d\mu(x)\,,$$

and so $\dfrac{f(x)}{1+x^2} = 1$ a.e., that is to say $f(x) = 1+x^2$ almost everywhere.

Problem 2.4.6 Let m be the Lebesgue measure on the real line $\mathbb{R}$. Consider a measurable function $f : \mathbb{R} \to [0,\infty]$ such that f and $1/f$ are Lebesgue integrable on each bounded subset of $\mathbb{R}$. For each Lebesgue measurable subset of $\mathbb{R}$ define

$$\mu(E) = \int_E f(x)\, dm(x)\,.$$

a) Show that $m \ll \mu$.
b) Compute the Radon-Nikodym derivative $h = dm/d\mu$.

Hint: Use the argument in the previous exercise.
Solution: b) $h(x) = 1/f(x)$.

Problem 2.4.7 Let us consider the increasing function $F(x) = \max\{0, x + [x]\}$, where $[x]$ denotes the integer part of x. Let μ_F be the Borel-Stieltjes measure associated to F.

a) Calculate $\mu_F((0,5])$, $\mu_F([4,8])$ and $\mu_F([3,7))$.
b) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure.

Hint: b) Consider, for example, the set $A = \{1\}$.
Solution: a) $\mu_F((0,5]) = 10$, $\mu_F([4,8]) = 9$, $\mu_F([3,7)) = 8$.
b) If $A = \{1\}$, we have that $\mu_F(A) = F(1+) - F(1-) = 2 - 1 = 1$ but $m(A) = 0$.

Problem 2.4.8 Let μ_F be the Borel-Stieltjes measure associated to the increasing function

$$F(x) = \begin{cases} 0\,, & \text{if } x < 0\,, \\ x\,, & \text{if } 0 \le x < 1\,, \\ 1\,, & \text{if } 1 \le x\,. \end{cases}$$

a) Prove that $\mu_F \ll m$, being m the Lebesgue measure.

b) Is there the Radon-Nikodym derivative of μ_F with respect to m?. If so, find it.

c) Let μ be the measure that counts rational numbers in $[0,1]$, that is to say that $\mu(A) = \operatorname{card}(A \cap [0,1] \cap \mathbb{Q})$. Prove that $\mu_F \perp \mu$.

Hints: a) Given $I = [a,b)$, prove that $\mu_F(I) \le m(I)$ since $x - F(x)$ is increasing. Apply Caratheodory-Hopf's theorem. b) Use problem 2.2.25. c) Consider the set $[0,1] \cap \mathbb{Q}$.

Solution: a) Since F is continuous we have that

$$\mu_F([a,b)) = F(b) - F(a) \le b - a = m([a,b))$$

because $F(x) \le x$, $\forall\, x \in \mathbb{R}$. In fact, the function $f(x) := x - F(x)$ is continuous and increasing. Hence, f define a Borel-Stieltjes measure μ_f and, by Caratheodory-Hopf's theorem we have that $\mu_f = m - \mu_F \ge 0$. Hence, $0 \le \mu \le m \implies \mu_F \ll m$.
b) Hence, there exists the Radon-Nikodym derivative of μ_F with respect to m. By problem 2.2.25 we have that

$$\mu_F = F'\,dm = \chi_{(0,1)}\,dm\,.$$

c) Let $B = [0,1] \cap \mathbb{Q}$. As $m(\mathbb{Q}) = 0$ we have that $m(B) = 0$ and, by part a) that $\mu_F(B) = 0$. On the other hand, $\mu(B^c) = \operatorname{card}(B^c \cap [0,1] \cap \mathbb{Q}) = \operatorname{card}(\varnothing) = 0$. Hence, $\mu_F \perp \mu$.

Problem 2.4.9 Let us define the increasing function

$$F(x) = \begin{cases} e^x\,, & \text{if } x < 0\,, \\ 2 + \arctan x\,, & \text{if } x \ge 0\,, \end{cases}$$

and let μ_F be the associated Borel-Stieltjes measure.

a) Calculate $\mu_F((0,1])$, $\mu_F((-2,0])$, $\mu_F(0)$ and $\mu_F(\mathbb{R})$.

b) Prove that $\mu_F = f(x)\,dx + \delta_0$, for certain $f \ge 0$, and being δ_0 the δ-Dirac measure at $x = 0$.

c) Is true that $\mu_F \ll dx$?

Hint: b) Observe that $\delta_0 = \mu_H$ with H the Heaviside function: $H = \chi_{[0,\infty)}$ and apply Exercise 2.2.25 to μ_{F-H}.

Solution: a) $\mu_f((0,1]) = \pi/4$, $\mu_f((-2,0]) = 2 - 1/e^2$, $\mu_f(0) = 1$, $\mu_f(\mathbb{R}) = 2 + \pi/2$. b) As $\delta_0 = \mu_H$ with H the Heaviside function $H(x) = 1$ for $x \ge 0$ and $H(x) = 0$ for $x < 0$, then

$$\mu_F - \delta_0 = \mu_F - \mu_H = \mu_{F-H}$$

and by problem 2.2.25 we have that

$$\mu_F - \delta_0 = f\,dx\,,\ \text{where}\ f(x) = (f-H)'(x) = \begin{cases} e^x\,, & \text{if } x<0\,,\\ \frac{1}{1+x^2}\,, & \text{if } x>0\,. \end{cases}$$

c) No, consider the set $E = \{0\}$. Then, $m(A) = 0$ but $\mu_F(A) = F(0^+) - F(0^-) = 2-1 = 1 \neq 0$.

Problem 2.4.10 Let us consider the increasing function

$$F(x) = \begin{cases} 0\,, & \text{if } x<0\,,\\ x^2/2\,, & \text{if } 0\leq x<1\,,\\ 1\,, & \text{if } 1\leq x\,, \end{cases}$$

and let μ_F be the Borel-Stieltjes measure associated to F.

a) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure m.

b) Find the Radon-Nikodym decomposition of μ_F with respect to m.

Hint: b) Use problem 2.2.25 on $(-\infty, 1)$ and on $(1,\infty)$.

Solution: a) Let $E = \{1\}$. Then, $m(E) = 0$ but $\mu_F(E) = F(1^+) - F(1^-) = 1 - 1/2 = 1/2 \neq 0$.
b) Using problem 2.2.25 we get that $\mu_F = f'\,dm + \lambda$ with $f(x) = F'(x) = x\chi_{(0,1)}(x)$, and $\lambda = \frac{1}{2}\,\delta_1$.