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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.2: Fourier transform

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2 Integrals depending on a parameter

3.2. Fourier transform

Problem 3.2.1 Prove that if $f \in L^1(\mathbb{R})$ and $f > 0$, then $|\hat{f}(\omega)| < \hat{f}(0)$ for every $\omega \neq 0$.

Hint: The inequality $|\hat{f}(\omega)| \leq \hat{f}(0)$ is easy. If α denotes the complex argument of $\hat{f}(\omega)$, then $|\hat{f}(\omega)| = \hat{f}(\omega)\, e^{-i\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i(\omega x-\alpha)} dx$. Now, take real parts in the equality $|\hat{f}(\omega)| = \hat{f}(0)$ to conclude that, a fortiori, $\omega = 0$.

Solution: First of all, as $f > 0$, we have that

$$|\hat{f}(\omega)| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|\,|e^{i\omega x}|\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, dx = \hat{f}(0)\,.$$

On the other hand, let $\hat{f}(\omega) = |\hat{f}(\omega)|\, e^{i\alpha}$ (α is the argument of the complex number $\hat{f}(\omega)$). Then

$$|\hat{f}(\omega)| = \hat{f}(\omega)\, e^{-i\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i(\omega x-\alpha)}\, dx\,.$$

If $|\hat{f}(\omega)| = \hat{f}(0)$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i(\omega x-\alpha)}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, dx\,.$$

Taking now real parts, we obtain that

$$\int_{-\infty}^{\infty} f(x)\cos(\omega x-\alpha)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

and so

$$\int_{-\infty}^{\infty} f(x)(1-\cos(\omega x-\alpha))\, dx = 0\,.$$

But $f(x)(1-\cos(\omega x-\alpha)) \geq 0$ for all x. Hence, we must have that

$$1-\cos(\omega x-\alpha) = 0 \;\; a.e.x \quad\Longrightarrow\quad \omega x-\alpha = 2\pi k \;\; a.e.x, \text{ for some } k\in\mathbb{Z} \quad\Longrightarrow\quad \omega = 0\,.$$

Problem 3.2.2 Given $\alpha > 0$, compute the Fourier transform of the following functions:

1) $f(x) = e^{-\alpha|x|}$,
2) $f(x) = \frac{2\alpha}{x^2+\alpha^2}$,
3) $f(x) = \chi_{[-\alpha,\alpha]}(x)$,
4) $f(x) = x\chi_{[-\alpha,\alpha]}(x)$,
5) $f(x) = \chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)$,
6) $f(x) = |x|\chi_{[-\alpha,\alpha]}(x)$,
7) $f(x) = \delta_0(x)$,
8) $f(x) = \frac{\sin\alpha x}{x}$,
9) $f(x) = (\alpha-|x|)\,\chi_{[-\alpha,\alpha]}$,
10) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
11) $f(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}$,
12) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
13) $f(x) = \frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}$,
14) $f(x) = \frac{1}{x}$,
15) $f(x) = \delta_{x_0} + \delta_{-x_0}$,
16) $f(x) = \delta_{x_0} - \delta_{-x_0}$,
17) $f(x) = e^{-\pi(x-3)^2}$,
18) $f(x) = e^{-i\pi(x+1)^2}$.

Solutions: 1) Applying directly the definition of the Fourier transform we obtain

$$\begin{aligned}\mathcal{F}\big[e^{-\alpha|x|}\big](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{i\omega x}dx = \frac{1}{2\pi}\int_{0}^{\infty} e^{-\alpha x}e^{i\omega x}dx + \frac{1}{2\pi}\int_{-\infty}^{0} e^{\alpha x}e^{i\omega x}dx \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{(i\omega-\alpha)x}dx + \frac{1}{2\pi}\int_{-\infty}^{0} e^{(i\omega+\alpha)x}dx \\ &= \frac{1}{2\pi}\Big(\Big[\frac{e^{(i\omega-\alpha)x}}{i\omega-\alpha}\Big]_{x=0}^{x=\infty} + \Big[\frac{e^{(i\omega+\alpha)x}}{i\omega+\alpha}\Big]_{x=-\infty}^{x=0}\Big) = \frac{1}{2\pi}\Big(\frac{-1}{i\omega-\alpha} + \frac{1}{i\omega+\alpha}\Big) \\ &= \frac{\alpha}{\pi(\omega^2+\alpha^2)}\ .\end{aligned}$$

2) Using the previous problem, we have:

$$\mathcal{F}^{-1}\Big[\frac{\alpha}{\pi(\omega^2+\alpha^2)}\Big](x) = e^{-\alpha|x|} \quad \Rightarrow \quad \mathcal{F}^{-1}\Big[\frac{\alpha}{\pi(x^2+\alpha^2)}\Big](\omega) = e^{-\alpha|\omega|}.$$

Taking this result into account and using the theorem on the inverse Fourier transform, we get

$$\mathcal{F}\Big[\frac{2\alpha}{x^2+\alpha^2}\Big](\omega) = \frac{1}{2\pi}\mathcal{F}^{-1}\Big[\frac{2\alpha}{x^2+\alpha^2}\Big](-\omega) = \mathcal{F}^{-1}\Big[\frac{\alpha}{\pi(x^2+\alpha^2)}\Big](-\omega) = e^{-\alpha|-\omega|} = e^{-\alpha|\omega|}.$$

3) Applying the definition of the Fourier transform we obtain

$$\begin{aligned}\mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi_{[-\alpha,\alpha]}(x)\,e^{i\omega x}dx = \frac{1}{2\pi}\int_{-\alpha}^{\alpha}e^{i\omega x}dx \\ &= \frac{1}{2\pi}\Big[\frac{e^{i\omega x}}{i\omega}\Big]_{x=-\alpha}^{x=\alpha} = \frac{e^{i\alpha\omega}-e^{-i\alpha\omega}}{2\pi i\omega} = \frac{\sin\alpha\omega}{\pi\omega}\ .\end{aligned}$$

4) As $\mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{\sin\alpha\omega}{\pi\omega}$ by the previous problem and the property 7 of the Fourier transform, we conclude that

$$\mathcal{F}\big[x\chi_{[-\alpha,\alpha]}(x)\big](\omega) = -i\,\frac{d}{d\omega}\big(\mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega)\big) = -i\,\frac{d}{d\omega}\Big(\frac{\sin\alpha\omega}{\pi\omega}\Big) = i\,\frac{\sin\alpha\omega-\alpha\omega\cos\alpha\omega}{\pi\omega^2}\ .$$

5) Applying the definition of the Fourier transform we obtain

$$\begin{aligned}\mathcal{F}\big[\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big)e^{i\omega x}dx \\ &= \frac{1}{2\pi}\int_{0}^{\alpha}e^{i\omega x}dx - \frac{1}{2\pi}\int_{-\alpha}^{0}e^{i\omega x}dx \\ &= \frac{1}{2\pi}\Big[\frac{e^{i\omega x}}{i\omega}\Big]_{x=0}^{x=\alpha} - \frac{1}{2\pi}\Big[\frac{e^{i\omega x}}{i\omega}\Big]_{x=-\alpha}^{x=0} = \frac{e^{i\alpha\omega}-1-1+e^{-i\alpha\omega}}{2\pi i\omega} \\ &= i\,\frac{1-\cos\alpha\omega}{\pi\omega}\ .\end{aligned}$$

6) As $\mathcal{F}\big[\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big](\omega) = i\,\frac{1-\cos\alpha\omega}{\pi\omega}$ by the previous problem and

$$|x|\chi_{[-\alpha,\alpha]}(x) = x\big(\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big)\ ,$$

the property 7 of the Fourier transform we conclude that

$$\begin{aligned}\mathcal{F}\big[|x|\chi_{[-\alpha,\alpha]}(x)\big](\omega) &= \mathcal{F}\big[x\big(\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big)\big](\omega) \\ &= -i\,\frac{d}{d\omega}\big(\mathcal{F}\big[\chi_{[0,\alpha]}(x)-\chi_{[-\alpha,0]}(x)\big](\omega)\big) = \frac{d}{d\omega}\Big(\frac{1-\cos\alpha\omega}{\pi\omega}\Big) \\ &= \frac{\alpha\omega\ \sin\alpha\omega+\cos\alpha\omega-1}{\pi\omega^2}\ .\end{aligned}$$

7) Applying the definitions of the Fourier transform and the Dirac delta, we obtain that

$$\mathcal{F}\big[\delta(x)\big](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)\, e^{i\omega x} dx = \frac{1}{2\pi}\, e^{i\omega x}\Big|_{x=0} = \frac{1}{2\pi}\,.$$

8) $\frac{1}{2}\,\chi_{[-\alpha,\alpha]}(\omega)$. 9) $\frac{1-\cos\alpha\omega}{\pi\omega^2}$. 10) $e^{-\alpha|\omega|}\cos x_0\omega$. 11) $e^{-i\alpha\omega^2}$. 12) $ie^{-\alpha|\omega|}\sin x_0\omega$.
13) $\frac{1}{2\alpha\beta(\alpha^2-\beta^2)}\big(\alpha e^{-\beta|\omega|} - \beta e^{-\alpha|\omega|}\big)$. 14) $-i/2$ if $\omega < 0$, 0 if $\omega = 0$, $i/2$ if $\omega > 0$. 15) $\frac{1}{\pi}\cos x_0\omega$.
16) $\frac{i}{\pi}\,\sin x_0\omega$. 17) $\frac{1}{2\pi}\,e^{i3\omega}\,e^{-\omega^2/(4\pi)}$. 18) $\frac{1}{2\pi}\,e^{-i(\omega+\pi/4)}\,e^{i\omega^2/(4\pi)}$.

Problem 3.2.3 Calculate the Fourier transform of the Gaussian function $f(x) = e^{-x^2}$.

Hint: Note that the imaginary part of $\hat{f}(\omega)$ is zero. To compute the real part use the theorem on derivation of parametric integrals $\big(\big|\frac{\partial}{\partial\omega}\big[e^{-x^2}\cos(\omega x)\big]\big| \le |x|e^{-x^2} \in L^1(\mathbb{R})\big)$. Integrating by parts prove that $\frac{d}{d\omega}[\hat{f}(\omega)] = -\frac{\omega}{2}\,\hat{f}(\omega)$. Recall that $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}\,$.
Solution: We have that

$$\mathcal{F}[e^{-x^2}](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\, e^{i\omega x}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2}\, \cos\omega x\, dx = \frac{1}{\pi}\int_{0}^{\infty} e^{-x^2}\, \cos\omega x\, dx\,,$$

since $\int_{-\infty}^{\infty} e^{-x^2}\,\sin\omega x\, dx = 0$ because $e^{-x^2}\,\sin\omega x$ is an odd function. Now, as

$$\Big|\frac{\partial}{\partial\omega}\big(e^{-x^2}\, \cos\omega x\big)\Big| = \big|e^{-x^2}\,(-x)\sin\omega x\big| \le x\, e^{-x^2} \in L^1(0,\infty)\,,$$

we can use the theorem on differentiation of parametric integrals obtaining

$$\frac{d}{d\omega}\big(\hat{f}(\omega)\big) = \frac{1}{\pi}\int_{0}^{\infty} \frac{\partial}{\partial\omega}\big(e^{-x^2}\, \cos\omega x\big)\, dx = \frac{-1}{\pi}\int_{0}^{\infty} x\, e^{-x^2}\, \sin\omega x\, dx\,.$$

Integrating by parts with $u = \sin\omega x$, $v' = xe^{-x^2}$, and using the dominated convergence theorem, we obtain that

$$\frac{d}{d\omega}\big(\hat{f}(\omega)\big) = \lim_{N\to\infty}\frac{1}{2\pi}\big[e^{-x^2}\, \sin\omega x\big]_{x=0}^{x=N} - \frac{\omega}{2\pi}\int_{0}^{\infty} e^{-x^2}\, \cos\omega x\, dx = -\frac{\omega}{2}\,\hat{f}(\omega)\,.$$

Hence, $\hat{f}'(\omega)/\hat{f}(\omega) = -\omega/2 \implies \log\hat{f}(\omega) = -\omega^2/4 + c \implies \hat{f}(\omega) = C\, e^{-\omega^2/4}$. But

$$\hat{f}(0) = \frac{1}{\pi}\int_{0}^{\infty} e^{-x^2}\, dx = \frac{1}{\pi}\,\frac{\sqrt{\pi}}{2} = \frac{1}{2\sqrt{\pi}} \qquad\Longrightarrow\qquad \hat{f}(\omega) = \frac{1}{2\sqrt{\pi}}\, e^{-\omega^2/4}\,.$$

Problem 3.2.4 For $\alpha > 0$, calculate the integral

$$\int_{-\infty}^{\infty}\frac{\sin^2\alpha x}{x^2}\, dx\,.$$

Hint: Use Plancherel's theorem and part 8) of Exercise 3.2.2.
Solution: Applying Plancherel's theorem and part 8) of Exercise 3.2.2 we obtain that

$$\int_{-\infty}^{\infty}\Big(\frac{\sin\alpha x}{x}\Big)^2 dx = 2\pi\int_{-\infty}^{\infty}\Big(\frac{1}{2}\,\chi_{[-\alpha,\alpha]}(\omega)\Big)^2 d\omega = \frac{\pi}{2}\int_{-\alpha}^{\alpha} d\omega = \alpha\pi\,.$$

Problem 3.2.5 Find a particular solution of the equation $u'' - u = f(x)$ by taking Fourier transforms in both sides of the equation.

Solution: Taking Fourier transforms in both members of the equation $u'' - u = f(x)$ we obtain that

$$-\omega^2 \mathcal{F}\big[u\big](\omega) - \mathcal{F}\big[u\big](\omega) = \mathcal{F}\big[f\big](\omega) \quad \Rightarrow \quad \mathcal{F}\big[u\big](\omega) = \frac{-1}{\omega^2+1}\, \mathcal{F}\big[f\big](\omega)\,.$$

As we know by the part 1) of problem 3.2.2. that $\mathcal{F}\big[e^{-|x|}\big](\omega) = 1/(\pi(\omega^2+1))$, we deduce using the property 6 on the Fourier transform of a convolution, that

$$\mathcal{F}\big[u\big](\omega) = -\pi \mathcal{F}\big[e^{-|x|}\big](\omega)\, \mathcal{F}\big[f\big](\omega) = -\pi \mathcal{F}\big[e^{-|x|} * f\big](\omega)\,,$$
$$u(x) = -\pi(e^{-|x|} * f)(x) = \frac{-1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy\,.$$

Problem 3.2.6 Find a solution of the initial value problem for the heat equation on $\mathbb{R}\times(0,\infty)$ by taking Fourier transforms in the x-variable in both members of the equations:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x\in\mathbb{R}\,,\ t>0\,,\\ u(x,0) \;=\; f(x)\,, & \text{if } x\in\mathbb{R}\,. \end{cases}$$

Solution: Let us denote by $U(\omega,t)$ and $F(\omega)$ the Fourier transforms in the variable x of the functions $u(x,t)$ and $f(x)$, respectively. Applying the Fourier transform in the variable x to both members of the equations, we obtain

$$\begin{cases} \frac{\partial}{\partial t}U(\omega,t) \;=\; -k\omega^2 U(\omega,t)\,,\\ U(\omega,0) \;=\; F(\omega)\,. \end{cases}$$

For each fixed ω, we can see the equation $\frac{\partial}{\partial t}U(\omega,t) = -k\omega^2 U(\omega,t)$ as an ordinary differential equation. The general solution of this equation is $U(\omega,t) = A\,e^{-k\omega^2 t}$, where A is a constant (with respect to the variable t, and so A can depend on the variable ω). Substituting the initial condition $U(\omega,0) = F(\omega)$ we obtain that $A = F(\omega)$ and so $U(\omega,t) = F(\omega)\,e^{-k\omega^2 t}$. If we define the function $K_t(x)$ through the following formula, using the result of problem 3.2.3 it is easy to obtain that:

$$K_t(x) = \sqrt{\frac{\pi}{kt}}\; e^{-x^2/(4kt)}, \qquad\qquad \mathcal{F}\big[K_t\big](\omega) \;=\; e^{-k\omega^2 t}.$$

Then, using the property on the Fourier transform of a convolution:

$$\mathcal{F}\big[u\big](\omega) = \mathcal{F}\big[K_t\big](\omega)\, \mathcal{F}\big[f\big](\omega) = \mathcal{F}\big[K_t * f\big](\omega)\,,$$
$$u(x,t) = (K_t * f)(x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\; dy\,.$$

Problem 3.2.7 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t) + c\,\frac{\partial}{\partial x}u(x,t)\,, & \text{if } x\in\mathbb{R}\,,\ t>0\,,\\ u(x,0) \;=\; f(x)\,, & \text{if } x\in\mathbb{R}\,. \end{cases}$$

Solution: We denote by $U(\omega,t)$ and $F(\omega)$ the Fourier transforms in the variable x of the functions $u(x,t)$ and $f(x)$, respectively. Applying the Fourier transform in the variable x to both members of the equations, we obtain

$$\begin{cases} \frac{\partial}{\partial t}U(\omega,t) = -k\,\omega^2 U(\omega,t) - i\,c\,\omega\, U(\omega,t)\,,\\ U(\omega,0) = F(\omega)\,. \end{cases}$$

For each fixed ω, we have the differential equation $\frac{\partial}{\partial t}U(\omega,t) = -(k\omega^2 + ic\omega)U(\omega,t)$, whose general solution is $U(\omega,t) = A\,e^{-(k\omega^2+ic\omega)t}$, where A is a constant (with respect to the variable t, and so A can depend on the variable ω). Substituting the initial condition $U(\omega,0) = F(\omega)$ we obtain that $A = F(\omega)$ and so $U(\omega,t) = F(\omega)\,e^{-k\omega^2 t}e^{-ict\omega}$. If we define the function $K_t(x)$ through the following expression (as in the previous problem), using the result of problem 3.2.3 it is easy to obtain that:

$$K_t(x) = \sqrt{\frac{\pi}{kt}}\; e^{-x^2/(4kt)}, \qquad\qquad \mathcal{F}\big[K_t\big](\omega) \;=\; e^{-k\omega^2 t}.$$

Hence, using the property 3 of the Fourier transform, we obtain $\mathcal{F}\big[K_t(x+ct)\big](\omega) \;=\; e^{-k\omega^2 t}e^{-ict\omega}$. Finally, using the property on the Fourier transform of a convolution, we get

$$\begin{aligned}\mathcal{F}\big[u\big](\omega) &= \mathcal{F}\big[K_t(x+ct)\big](\omega)\,\mathcal{F}\big[f\big](\omega) = \mathcal{F}\big[K_t(x+ct)*f\big](\omega)\,,\\ u(x,t) &= (K_t(x+ct)*f)(x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x+ct-y)^2/(4kt)}\, f(y)\;dy\,.\end{aligned}$$

Problem 3.2.8 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases}\frac{\partial}{\partial t}u(x,t) \;=\; \frac{\partial^2}{\partial x^2}u(x,t) - 2\,\frac{\partial}{\partial x}u(x,t)\,, & \text{if } x\in\mathbb{R}\,,\ t>0\,,\\ u(x,0) \;=\; e^{-x^2}, & \text{if } x\in\mathbb{R}\,.\end{cases}$$

Solution: Using the previous problem we know that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-2t-y)^2/(4t)}e^{-y^2}\;dy = \frac{e^{-(x-2t)^2/(4t)}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-[(1+4t)y^2-2(x-2t)y]/(4t)}\;dy\,.$$

As

$$\begin{aligned}(1+4t)y^2-2(x-2t)y &= (1+4t)\Big(y^2 - 2\,\frac{x-2t}{1+4t}\,y + \frac{(x-2t)^2}{(1+4t)^2} - \frac{(x-2t)^2}{(1+4t)^2}\Big)\\ &= (1+4t)\Big(y-\frac{x-2t}{1+4t}\Big)^2 - \frac{(x-2t)^2}{1+4t}\;.\end{aligned}$$

We have with the change of variables $v = y-(x-2t)/(1+4t)$ and $w = v\sqrt{1+4t}/\sqrt{4t}$, and using again the problem 3.2.3 that

$$\begin{aligned}u(x,t) &= \frac{e^{-(x-2t)^2/(4t)}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(1+4t)\big(y-(x-2t)/(1+4t)\big)^2/(4t)}e^{(x-2t)^2/(4t(1+4t))}\;dy\\ &= \frac{e^{-(x-2t)^2/(1+4t)}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(1+4t)v^2/(4t)}\;dv\\ &= \frac{e^{-(x-2t)^2/(1+4t)}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-w^2}\,\frac{\sqrt{4t}}{\sqrt{1+4t}}\;dw = \frac{1}{\sqrt{1+4t}}\;e^{-(x-2t)^2/(1+4t)}.\end{aligned}$$

Problem 3.2.9 Find a solution of the initial value problem for the diffusion equation with absorption:

$$\begin{cases}\frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t) - c\,u(x,t)\,, & \text{if } x\in\mathbb{R}\,,\ t>0\,,\\ u(x,0) \;=\; f(x)\,, & \text{if } x\in\mathbb{R}\,.\end{cases}$$

Solution: We denote by $U(\omega, t)$ and $F(\omega)$ the Fourier transforms in the variable x of the functions $u(x,t)$ and $f(x)$, respectively. Applying the Fourier transform in the variable x to both members of the equations, we obtain

$$\begin{cases} \frac{\partial}{\partial t}U(\omega,t) = -k\,\omega^2 U(\omega,t) - c\,U(\omega,t)\,, \\ U(\omega,0) = F(\omega)\,. \end{cases}$$

For each fixed ω, we have the ordinary differential equation $\frac{\partial}{\partial t}U(\omega,t) = -(k\omega^2+c)U(\omega,t)$, whose general solution is $U(\omega,t) = A\,e^{-(k\omega^2+c)t}$, where A is a constant (with respect to the variable t, and so A can depend on thea variable ω). Substituting the initial condition $U(\omega,0) = F(\omega)$ we obtain that $A = F(\omega)$ and so $U(\omega,t) = e^{-ct}F(\omega)\,e^{-k\omega^2 t}$. If we define the function $K_t(x)$ through the following expression, as in the previous problems, using the result of problem 3.2.3 it is easy to obtain that:

$$K_t(x) = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/(4kt)}, \qquad \mathcal{F}\big[K_t\big](\omega) \;=\; e^{-k\omega^2 t}.$$

Then using the property on the Fourier transform of a convolution, we deduce that

$$\mathcal{F}\big[u\big](\omega) = e^{-ct}\mathcal{F}\big[K_t\big](\omega)\,\mathcal{F}\big[f\big](\omega) = e^{-ct}\mathcal{F}\big[K_t * f\big](\omega)\,,$$
$$u(x,t) = e^{-ct}(K_t * f)(x) = \frac{e^{-ct}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)}\, f(y)\; dy\,.$$

Problem 3.2.10 Find the solution of the initial value problem for the wave equation on $\mathbb{R}\times(0,\infty)$:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) \;=\; c^2\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x\in\mathbb{R}\,,\ t>0\,, \\ u(x,0) \;=\; f(x)\,, & \text{if } x\in\mathbb{R}\,, \\ \frac{\partial}{\partial t}u(x,0) \;=\; g(x)\,, & \text{if } x\in\mathbb{R}\,. \end{cases}$$

Solution: Let us denote by $U(\omega,t)$, $F(\omega)$ and $G(\omega)$ the Fourier transforms in the variable x of the functions $u(x,t)$, $f(x)$ and $g(x)$, respectively. Applying the Fourier transform in the variable x to both members of the equations, we obtain that

$$\begin{cases} \frac{\partial^2}{\partial t^2}U(\omega,t) \;=\; -c^2\omega^2 U(\omega,t)\,, \\ U(\omega,0) \;=\; F(\omega)\,, \\ \frac{\partial}{\partial t}U(\omega,0) \;=\; G(\omega)\,. \end{cases}$$

For each fixed ω, we have the ordinary differential equation $\frac{\partial^2}{\partial t^2}U(\omega,t) = -c^2\omega^2 U(\omega,t)$, whose general solution is $U(\omega,t) = A\cos(c\omega t) + B\,\sin(c\omega t)$, where A and B are constants (with respect to the variable t, and so A and B can depend on the variable ω). Substituting the initial conditions $U(\omega,0) = F(\omega)$ and $\frac{\partial}{\partial t}U(\omega,0) = G(\omega)$ we obtain that $A = F(\omega)$ and $B = G(\omega)/(c\omega)$; Hence, $U(\omega,t) = F(\omega)\cos(c\omega t) + G(\omega)\,\frac{\sin(c\omega t)}{c\omega}$.
If we define the function $E_t(x)$ through the following expression, the part 3 of problem 3.2.2 gives:

$$E_t(x) = \frac{\pi}{c}\,\chi_{[-ct,ct]}(x)\,, \qquad \mathcal{F}\big[E_t(x)\big](\omega) = \frac{\sin(c\omega t)}{c\omega}\,.$$

From this last equality and property 9 of the Fourier transform we deduce

$$\mathcal{F}\Big[\frac{\partial E_t}{\partial t}\Big](\omega) = \frac{\partial}{\partial t}\big(\mathcal{F}\big[E_t\big](\omega)\big) = \frac{\partial}{\partial t}\Big(\frac{\sin(c\omega t)}{c\omega}\Big) = \cos(c\omega t)\,.$$

Then, using the linearity of the Fourier transform and the property on the Fourier transform of a convolution, we get

$$\mathcal{F}\big[u\big](\omega) = \mathcal{F}\Big[\frac{\partial E_t}{\partial t}\Big](\omega)\,\mathcal{F}\big[f\big](\omega) + \mathcal{F}\big[E_t\big](\omega)\,\mathcal{F}\big[g\big](\omega) = \mathcal{F}\Big[\frac{\partial E_t}{\partial t} * f + E_t * g\Big](\omega)\,,$$
$$u(x,t) = \Big(\frac{\partial E_t}{\partial t} * f\Big)(x) + \big(E_t * g\big)(x) = \frac{\partial}{\partial t}\big(E_t * f\big)(x) + \big(E_t * g\big)(x)\,.$$

As

$$\big(E_t * g\big)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-y)\,\frac{\pi}{c}\,\chi_{[-ct,ct]}(y)\,dy = \frac{1}{2c}\int_{-ct}^{ct} g(x-y)\,dy = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\,,$$
$$\big(E_t * f\big)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} f(s)\,ds\,, \qquad \frac{\partial}{\partial t}\big(E_t * f\big)(x) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big)\,,$$

we obtain that

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\;ds\,.$$

This expression is known as <u>*D'Alembert's formula*</u>.

Problem 3.2.11 Prove that if f is of C^2-class (continuous with two continuous derivatives) on $\mathbb{R}$ and g is of C^1-class (continuous with one continuous derivative) on $\mathbb{R}$, then D'Alembert's formula

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\;ds\,,$$

which has been obtained in the previous problem, is effectively a solution of the initial value problem for the wave equation on $\mathbb{R} \times (0,\infty)$.

Solution: As f belongs to the class C^2 and g to the class C^1, we have that

$$\frac{\partial u}{\partial x}(x,t) = \frac{1}{2}\big(f'(x+ct) + f'(x-ct)\big) + \frac{1}{2c}\big(g(x+ct) - g(x-ct)\big)\,,$$
$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{1}{2}\big(f''(x+ct) + f''(x-ct)\big) + \frac{1}{2c}\big(g'(x+ct) - g'(x-ct)\big)\,,$$
$$\frac{\partial u}{\partial t}(x,t) = \frac{c}{2}\big(f'(x+ct) - f'(x-ct)\big) + \frac{1}{2}\big(g(x+ct) + g(x-ct)\big)\,,$$
$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{c^2}{2}\big(f''(x+ct) + f''(x-ct)\big) + \frac{c}{2}\big(g'(x+ct) - g'(x-ct)\big)\,,$$

and so,

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\frac{\partial^2 u}{\partial x^2}(x,t)\,.$$

Substituting $t = 0$ in $u(x,t)$ and $\frac{\partial}{\partial t}u(x,t)$ we get

$$u(x,0) = \frac{1}{2}\big(f(x) + f(x)\big) + \frac{1}{2c}\int_{x}^{x} g(s)\;ds = f(x)\,,$$
$$\frac{\partial u}{\partial t}(x,0) = \frac{c}{2}\big(f'(x) - f'(x)\big) + \frac{1}{2}\big(g(x) + g(x)\big) = g(x)\,.$$

Hence, D'Alembert's formula provides a solution of the initial value problem for the wave equation on $\mathbb{R} \times (0,\infty)$.

Problem 3.2.12 Find the solution of the initial value problem for the non-homogeneous wave equation on $\mathbb{R} \times \mathbb{R}$:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) \;=\; \frac{\partial^2}{\partial x^2} u(x,t) + 6\,, & \text{if } x \in \mathbb{R}\,,\, t \in \mathbb{R}\,, \\ u(x,0) \;=\; x^2\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t} u(x,0) \;=\; 4x\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Solution: It is easy to check that $u_0(x,t) = 3t^2$ is a particular solution of the non-homogeneous equation $\frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + 6$, since $\frac{\partial^2}{\partial t^2} u(x,t) = 6$ and $\frac{\partial^2}{\partial x^2} u(x,t) = 0$.
It is also easy to see that the function v defined as $v(x,t) = u(x,t) - u_0(x,t) = u(x,t) - 3t^2$ is a solution of the initial value problem for the homogeneous wave equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2} v(x,t) = \frac{\partial^2}{\partial x^2} v(x,t)\,, & \text{if } x \in \mathbb{R}\,,\, t > 0\,, \\ v(x,0) = u(x,0) - u_0(x,0) = x^2\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t} v(x,0) = \frac{\partial}{\partial t} u(x,0) - \frac{\partial}{\partial t} u_0(x,0) = 4x\,, & \text{if } x \in \mathbb{R}\,, \end{cases}$$

since

$$\begin{aligned} & \frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + 6 \\ \Rightarrow \quad & \frac{\partial^2 v}{\partial t^2} v(x,t) + \frac{\partial^2 u_0}{\partial t^2}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t) + \frac{\partial^2 u_0}{\partial x^2}(x,t) + 6 \\ \Rightarrow \quad & \frac{\partial^2 v}{\partial t^2} v(x,t) + 6 = \frac{\partial^2 v}{\partial x^2}(x,t) + 6 \quad \Rightarrow \quad \frac{\partial^2 v}{\partial t^2} v(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t)\,. \end{aligned}$$

Hence, D'Alembert's formula (see previous problems) gives

$$\begin{aligned} v(x,t) &= \frac{1}{2}\big(f(x+t) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ &= \frac{1}{2}\big((x+t)^2 + (x-t)^2\big) + \frac{1}{2}\int_{x-t}^{x+t} 4s\,ds \\ &= x^2 + t^2 + \big[s^2\big]_{s=x-t}^{s=x+t} = x^2 + t^2 + 4xt\,. \end{aligned}$$

Then our solution u is

$$u(x,t) = v(x,t) + u_0(x,t) = x^2 + 4t^2 + 4xt = (x+2t)^2\,.$$

FOURIER TRANSFORMS TABLE $(x_0 \in \mathbb{R}, \alpha, \beta > 0)$

$$(TF1) \quad \mathcal{F}\big[e^{-\alpha x^2}\big](\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/(4\alpha)},$$

$$(TF2) \quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/(4\alpha)}\Big](\omega) = e^{-\alpha\omega^2},$$

$$(TF3) \quad \mathcal{F}\big[e^{-\alpha|x|}\big](\omega) = \frac{\alpha}{\pi(\omega^2+\alpha^2)}\,,$$

$$(TF4) \quad \mathcal{F}\Big[\frac{2\alpha}{x^2+\alpha^2}\Big](\omega) = e^{-\alpha|\omega|},$$

$$(TF5) \quad \mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{\sin\alpha\omega}{\pi\omega}\,, \qquad \text{if } \chi_{[a,b]}(x) = \begin{cases} 1\,, & \text{if } x \in [a,b]\,, \\ 0\,, & \text{if } x \notin [a,b]\,, \end{cases}$$

$$(TF6) \quad \mathcal{F}\Big[\frac{\sin\alpha x}{x}\Big](\omega) = \frac{1}{2}\,\chi_{[-\alpha,\alpha]}(\omega)\,,$$

$$(TF7) \quad \mathcal{F}\big[x\chi_{[-\alpha,\alpha]}(x)\big](\omega) = i\,\frac{\sin\alpha\omega - \alpha\omega\cos\alpha\omega}{\pi\omega^2}\,,$$

$$(TF8) \quad \mathcal{F}\big[\chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)\big](\omega) = i\,\frac{1-\cos\alpha\omega}{\pi\omega}\,,$$

$$(TF9) \quad \mathcal{F}\big[|x|\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{\alpha\omega\,\sin\alpha\omega + \cos\alpha\omega - 1}{\pi\omega^2}\,,$$

$$(TF10) \quad \mathcal{F}\big[(\alpha-|x|)\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{1-\cos\alpha\omega}{\pi\omega^2} = \frac{\sin^2(\alpha\omega/2)}{2\pi\omega^2}\,,$$

$$(TF11) \quad \mathcal{F}\big[e^{-i\alpha x^2}\big](\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-i\pi/4}\, e^{i\omega^2/(4\alpha)},$$

$$(TF12) \quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}\Big](\omega) = e^{-i\alpha\omega^2},$$

$$(TF13) \quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) = e^{-\alpha|\omega|}\cos x_0\omega\,,$$

$$(TF14) \quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) = ie^{-\alpha|\omega|}\,\sin x_0\omega\,,$$

$$(TF15) \quad \mathcal{F}\Big[\frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}\Big](\omega) = \frac{1}{2\alpha\beta(\alpha^2-\beta^2)}\big(\alpha e^{-\beta|\omega|} - \beta e^{-\alpha|\omega|}\big)\,,$$

$$(TF16) \quad \mathcal{F}\Big[\frac{1}{x}\Big](\omega) = \begin{cases} -i/2\,, & \text{if } \omega < 0\,, \\ 0\,, & \text{if } \omega = 0\,, \\ i/2\,, & \text{if } \omega > 0\,, \end{cases} \qquad \text{(it's understood as the principal value)},$$

$$(TF17) \quad \mathcal{F}\big[\delta_0\big](\omega) = \frac{1}{2\pi}\,, \qquad \mathcal{F}\big[\delta_{x_0}\big](\omega) = \frac{1}{2\pi}\, e^{ix_0\omega},$$

$$(TF18) \quad \mathcal{F}\big[\delta_{x_0} + \delta_{-x_0}\big](\omega) = \frac{1}{\pi}\cos x_0\omega\,,$$

$$(TF19) \quad \mathcal{F}\big[\delta_{x_0} - \delta_{-x_0}\big](\omega) = \frac{i}{\pi}\,\sin x_0\omega\,.$$