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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.3: Laplace transform

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

2 Integrals depending on a parameter

3.3. Laplace transform

Problem 3.3.1 Given $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, calculate the Laplace's transform of the following functions:

1) $f(t) = 1$,	2) $f(t) = t^n$,
3) $f(t) = \cos at$,	4) $f(t) = \sin at$,
5) $f(t) = \frac{\sin at}{t}$,	6) $f(t) = e^{at}$,
7) $f(t) = t^n e^{at}$,	8) $f(t) = e^{at} \sin bt$,
9) $f(t) = e^{at} \cos bt$,	10) $f(t) = e^{at} \sinh bt$,
11) $f(t) = e^{at} \cosh bt$,	12) $f(t) = t \cos at$,
13) $f(t) = t^2 e^{-t} \cos t$,	14) $f(t) = t \int_0^t e^{-x} \sin x \, dx$,
15) $f(t) = \cos^3 t$,	16) $f(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1, \\ (t-1)^k, & \text{if } t > 1, \end{cases} \quad (k \in \mathbb{N})$.

Solution: 1) Integrating directly:

$$L[1](z) = \int_0^\infty e^{-zt} dt = \lim_{N \to \infty} \left[\frac{e^{-zt}}{-z} \right]_{t=0}^{t=N} = \lim_{N \to \infty} \frac{1 - e^{-zN}}{z} = \frac{1}{z} .$$

2) We can calculate it integrating by parts and applying induction, but we can also apply the property (7) of the Laplace transform:

$$L[t^n](z) = (-1)^n \frac{d^n}{dz^n} \big(L[1](z) \big) = (-1)^n \frac{d^n}{dz^n} \left(\frac{1}{z} \right) = \frac{n!}{z^{n+1}} .$$

The last identity can be proved by induction: it is clearly true for $n = 1$; if we suppose that $(d^n/dz^n)(1/z) = (-1)^n n!/z^{n+1}$, then $(d^{n+1}/dz^{n+1})(1/z) = (d/dz)((-1)^n n!/z^{n+1}) = (-1)^{n+1}(n+1)!/z^{n+2}$.

3) and 4): We can calculate them integrating by parts, but we are going to obtain them as an application of property (5) of the Laplace transform. Applying this result to the sine and cosine functions we obtain:

$$L[\cos t](z) = z \, L[\sin t](z) , \qquad -L[\sin t](z) = z \, L[\cos t](z) - 1 .$$

Substituting the second equation in the first one, we obtain that:

$$L[\cos t](z) = -z^2 \, L[\cos t](z) + z \quad \Longrightarrow \quad L[\cos t](z) = \frac{z}{z^2 + 1} ,$$

and substituting, for example, in the first equation:

$$L[\sin t](z) = \frac{1}{z} \, L[\cos t](z) = \frac{1}{z^2 + 1} .$$

Finally, the property (3) of the Laplace transform:

$$L[\cos at](z) = \frac{1}{a} \, L[\cos t](z/a) = \frac{1}{a} \, \frac{z/a}{(z/a)^2 + 1} = \frac{z}{z^2 + a^2} ,$$
$$L[\sin at](z) = \frac{1}{a} \, L[\sin t](z/a) = \frac{1}{a} \, \frac{1}{(z/a)^2 + 1} = \frac{a}{z^2 + a^2} .$$

5) Applying the property (9) of the Laplace transform:

$$L\Big[\frac{\sin at}{t}\Big](z) = \int_z^\infty L[\sin at](\tau)\,d\tau = \int_z^\infty \frac{a}{\tau^2+a^2}\,d\tau = \lim_{N\to\infty}\Big[\arctan\frac{\tau}{a}\Big]_{\tau=z}^{\tau=N}$$
$$= \lim_{N\to\infty}\arctan\frac{N}{a} - \arctan\frac{z}{a} = \frac{\pi}{2} - \arctan\frac{z}{a} = \arctan\frac{a}{z}\,.$$

6) Using the property (2) of the Laplace transform:

$$L[e^{at}](z) = L[1](z-a) = \frac{1}{z-a}\,, \qquad \text{if } \operatorname{Re} z > a\,.$$

7) Using again the property (2) of the Laplace transform:

$$L[t^n e^{at}](z) = L[t^n](z-a) = \frac{n!}{(z-a)^{n+1}}\,, \qquad \text{if } \operatorname{Re} z > a\,.$$

8) and 9): Again by the property (2) of the Laplace transform:

$$L[e^{at}\cos bt](z) = L[\cos bt](z-a) = \frac{z-a}{(z-a)^2+b^2}\,,$$
$$L[e^{at}\sin bt](z) = L[\sin bt](z-a) = \frac{b}{(z-a)^2+b^2}\,.$$

10) and 11): First, by the linearity of the Laplace transform:

$$L[\cosh bt](z) = L[(e^{bt}+e^{-bt})/2](z) = \frac{1}{2}(L[e^{bt}](z) + L[e^{-bt}](z)) = \frac{1}{2}\Big(\frac{1}{z-b}+\frac{1}{z+b}\Big) = \frac{z}{z^2-b^2}\,,$$
$$L[\sinh bt](z) = L[(e^{bt}-e^{-bt})/2](z) = \frac{1}{2}(L[e^{bt}](z) - L[e^{-bt}](z)) = \frac{1}{2}\Big(\frac{1}{z-b}-\frac{1}{z+b}\Big) = \frac{b}{z^2-b^2}\,,$$

and using now the property (2) of the Laplace transform:

$$L[e^{at}\cosh bt](z) = L[\cosh bt](z-a) = \frac{z-a}{(z-a)^2-b^2}\,,$$
$$L[e^{at}\sinh bt](z) = L[\sinh bt](z-a) = \frac{1}{(z-a)^2-b^2}\,.$$

12) Using the property (7) of the Laplace transform:

$$L[t\cos at](z) = -\frac{d}{dz}\big(L[\cos at](z)\big) = -\frac{d}{dz}\Big(\frac{z}{z^2+a^2}\Big) = \frac{z^2-a^2}{(z^2+a^2)^2}\,.$$

13) Using again the property (7) of the Laplace transform and the part 9) of this problem:

$$L[t^2e^{-t}\cos t](z) = \frac{d^2}{dz^2}\big(L[e^{-t}\cos t](z)\big) = \frac{d^2}{dz^2}\Big(\frac{z+1}{(z+1)^2+1}\Big) = -2\,\frac{(z+1)(3-(z+1)^2)}{((z+1)^2+1)^3}\,.$$

14) Using the properties (7) and (8) of the Laplace transform and the part 8) of this problem:

$$L\Big[t\int_0^t e^{-x}\sin x\,dx\Big](z) = -\frac{d}{dz}\Big(L\Big[\int_0^t e^{-x}\sin x\,dx\Big]\Big)(z) = -\frac{d}{dz}\Big(\frac{1}{z}\,L[e^{-t}\sin t](z)\Big)$$
$$= -\frac{d}{dz}\Big(\frac{1}{z((z+1)^2+1)}\Big) = \frac{3z^2+4z+2}{(z^3+2z^2+2z)^2}\,.$$

15) Fist, we must express $f(t)$ in terms of simpler functions. As

$$\begin{aligned}\cos 3x + i\sin 3x = e^{3ix} = (e^{ix})^3 = (\cos x + i\sin x)^3\\ = \cos^3 x + 3i\cos^2 x \sin x - 3\cos x \sin^2 x - i\sin^3 x\end{aligned}$$

we deduce that $\cos^3 x = \cos 3x + 3\cos x \sin^2 x = \cos 3x + 3\cos x(1-\cos^2 x) = \cos 3x + 3\cos x - 3\cos^3 x$, and so $4\cos^3 x = \cos 3x + 3\cos x$, that is to say,

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x\,.$$

Therefore, using now the part 3) of this problem:

$$L[\cos^3 t](z) = \frac{1}{4}L[\cos 3t](z) + \frac{3}{4}L[\cos t](z) = \frac{1}{4}\frac{z}{z^2+9} + \frac{3}{4}\frac{z}{z^2+1} = \frac{z^3+7z}{(z^2+1)(z^2+9)}\,.$$

16) Let us observe that $f(t) = (t-1)^k H(t-1)$ where $H(t)$ is the Heaviside function. Hence, using the property (4) of the Laplace transform we get

$$L[f](z) = e^{-z}L[t^k](z) = \frac{k!}{z^{k+1}e^z}\,.$$

Problem 3.3.2 a) A function f is said periodic with period $P > 0$, if $f(t) = f(P+t)$ for all $t \in (0,\infty)$. Prove that the Laplace transform transform of a periodic function with period $P > 0$ verifies

$$\mathcal{L}[f](z) = \frac{1}{1-e^{-Pz}}\int_0^P e^{-zt} f(t)\,dt\,.$$

b) Calculate the Laplace transform of the function $f(t) = t - [t]$, being $[t]$ the integer part of t.

Solution: a) Decomposing the integral into two parts and making the change of variable $t = u+P$ in the second one, we obtain that:

$$L[f](z) = \int_0^P e^{-zt} f(t)\,dt + \int_P^\infty e^{-zt} f(t)\,dt = \int_0^P e^{-zt} f(t)\,dt + \int_0^\infty e^{-z(u+P)} f(u+P)\,du\,.$$

But, as $f(u+P) = f(u)$,

$$L[f](z) = \int_0^P e^{-zt} f(t)\,dt + e^{-Pz}\int_0^\infty e^{-zu} f(u)\,du = \int_0^P e^{-zt} f(t)\,dt + e^{-Pz}L[f](z)\,,$$

from which the requested formula is deduced after solving for $L[f](s)$.

b) Let us observe that $f(t)$ is the decimal part of t and so it is periodic with period $P = 1$. Hence, using the part a) we get

$$L[f](z) = \frac{1}{1-e^{-z}}\int_0^1 e^{-zt} t\,dt\,.$$

Integrating by parts, we obtain, for $\operatorname{Re} z > 0$, that:

$$\begin{aligned}L[f](z) &= \frac{1}{1-e^{-z}}\Big(\Big[-\frac{1}{z}\,t\,e^{-zt}\Big]_{t=0}^{t=1} + \frac{1}{z}\int_0^1 e^{-zt}dt\Big) = \frac{1}{1-e^{-z}}\Big(-\frac{e^{-z}}{z} - \frac{1}{z^2}\left[e^{-zt}\right]_{t=0}^{t=1}\Big)\\ &= \frac{1}{1-e^{-z}}\Big(-\frac{e^{-z}}{z} - \frac{1}{z^2}(e^{-z}-1)\Big) = \frac{1}{z^2} - \frac{1}{z(e^z-1)}\,.\end{aligned}$$

Problem 3.3.3 Calculate the inverse Laplace transform of the following functions:

$$1)\ F(z) = \frac{e^{3z}}{z^3 + 4z}\,, \qquad 2)\ F(z) = \frac{e^{-3z}}{z^3 + 4z}\,,$$
$$3)\ F(z) = \frac{1}{(z^2 + a^2)^2}\,, \qquad 4)\ F(z) = \frac{z}{(z^2 + a^2)^2}\,,$$
$$5)\ F(z) = \log(1 + 1/z)\,.$$

Solution: 1) It does not exist since $\lim_{\mathrm{Re}(z)\to\infty} F(z) \neq 0$.
2) By the property (4) of the Laplace transform:

$$F(z) = L[f(t-3)\,H(t-3)](z)\,, \qquad \text{with } L[f](z) = \frac{1}{z^3 + 4z}\,.$$

To obtain $f(t)$ we can use several methods:
a) Decomposing into simple functions: $\dfrac{1}{z^3+4z} = \dfrac{1}{4}\dfrac{1}{z} - \dfrac{1}{4}\dfrac{z}{z^2+4}$, and using the table of Laplace transforms:

$$f(t) = L^{-1}\Big[\frac{1}{z^3+4z}\Big](t) = \frac{1}{4}\,L^{-1}\Big[\frac{1}{z}\Big](t) - \frac{1}{4}\,L^{-1}\Big[\frac{z}{z^2+4}\Big](t) = \frac{1}{4}\,(1 - \cos 2t) = \frac{1}{2}\,\sin^2 t\,.$$

b) Using the convolution for Laplace transform (See Definition 3.6 and Proposition 3.7):

$$\begin{aligned} f(t) &= L^{-1}\Big[\frac{1}{z^3+4z}\Big](t) = L^{-1}\Big[\frac{1}{z}\,\frac{1}{z^2+4}\Big](t) = \Big(L^{-1}\Big[\frac{1}{z}\Big] * L^{-1}\Big[\frac{1}{z^2+4}\Big]\Big)(t) \\ &= (1 * (\sin 2t)/2)(t) = \frac{1}{2}\int_0^t \sin 2(t-x)\,dx = \frac{1}{4}\,\big[\cos 2(t-x)\big]_{x=0}^{x=t} = \frac{1}{4}\,(1 - \cos 2t)\,. \end{aligned}$$

Once we have obtained $f(t)$ applying the property (4) of the Laplace transform, we deduce that:

$$L^{-1}[F](t) = \frac{1}{2}\,\sin^2(t-3)\,H(t-3)\,.$$

3) Using the table of Laplace transforms, we know that $L[\sin at](z) = a/(z^2+a^2)$. Hence, $F(z) = L[\frac{1}{a}\sin at](z)\cdot L[\frac{1}{a}\sin at](z)$ and so, by the Proposition 3.7 on the Laplace transform of the convolution, we deduce that $F = L[f]$ with

$$f(t) = \Big(\frac{1}{a}\sin at\Big) * \Big(\frac{1}{a}\sin at\Big) = \frac{1}{a^2}\int_0^t \sin ax \sin[a(t-x)]\,dx\,.$$

Using now the trigonometric formula $2\sin A \sin B = \cos(A-B) - \cos(A+B)$, we get

$$\begin{aligned} f(t) &= \frac{1}{2a^2}\int_0^t \big[\cos\big(ax - a(t-x)\big) - \cos\big(ax + a(t-x)\big)\big]\,dx \\ &= \frac{1}{2a^2}\int_0^t \big[\cos(2ax - at) - \cos at\big]\,dx = \frac{1}{2a^2}\Big[\frac{\sin(2ax-at)}{2a} - x\cos at\Big]_{x=0}^{x=t} \\ &= \frac{1}{2a^2}\Big[\frac{\sin at}{2a} - t\cos at - \frac{\sin(-at)}{2a}\Big] = \frac{1}{2a^3}\sin at - \frac{1}{2a^2}\,t\cos at\,. \end{aligned}$$

4) Using the table of Laplace transforms, we know that $L[\sin at](z) = a/(z^2+a^2)$ and $L[\cos at](z) = z/(z^2+a^2)$. Hence, $F(z) = L[\frac{1}{a}\sin at](z)\cdot L[\cos at](z)$ and so, by the Proposition 3.7 on the Laplace transform of the convolution, we deduce that $F = L[f]$ with

$$f(t) = \Big(\frac{1}{a}\sin at\Big) * \Big(\cos at\Big) = \frac{1}{a}\int_0^t \sin ax \cos[a(t-x)]\,dx\,.$$

Using now the trigonometric formula $2\sin A\cos B = \sin(A+B)+\sin(A-B)$, we get

$$\begin{aligned} f(t) &= \frac{1}{2a}\int_0^t \big[\sin\big(ax+a(t-x)\big)+\sin\big(ax-a(t-x)\big)\big]\,dx \\ &= \frac{1}{2a}\int_0^t \big[\sin(2ax-at)+\sin at\big]\,dx = \frac{1}{2a}\Big[\frac{-\cos(2ax-at)}{2a}+x\sin at\Big]_{x=0}^{x=t} \\ &= \frac{1}{2a}\Big[\frac{-\cos at}{2a}+t\sin at-\frac{-\cos(-at)}{2a}\Big] = \frac{1}{2a}\,t\sin at\,. \end{aligned}$$

5) Using the property (7) of Laplace transform we have that $F'(z) = -L[tf(t)](z)$. But, $F'(z) = 1/(z+1) - 1/z$, and so we deduce that $-tf(t) = e^{-t}-1$. Hence, $f(t) = (1-e^{-t})/t$.

Problem 3.3.4 a) If $F(z)$ denotes the Laplace transform of f and $f \in L^1(0,\infty)$, prove the identity

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(s)\,ds\,.$$

b) Use this identity to calculate

$$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\,dx\,.$$

Solution: a) Using Fubini's theorem we obtain that:

$$\begin{aligned} \int_0^\infty F(s)\,ds &= \int_0^\infty\int_0^\infty f(t)\,e^{-st}dt\,ds = \int_0^\infty \Big(\int_0^\infty e^{-st}ds\Big) f(t)\,dt \\ &= \int_0^\infty \Big(\lim_{N\to\infty}\Big[-\frac{e^{-st}}{t}\Big]_{s=0}^{s=N}\Big) f(t)\,dt = \int_0^\infty \frac{f(t)}{t}\,dt\,. \end{aligned}$$

b) Taking $f(t) = e^{-\alpha t}-e^{-\beta t}$ in part a), we obtain that:

$$\begin{aligned} \int_0^\infty \frac{e^{-\alpha t}-e^{-\beta t}}{t}\,dt &= \int_0^\infty L[e^{-\alpha t}-e^{-\beta t}](s)\,ds \\ &= \int_0^\infty \Big(\frac{1}{s+\alpha}-\frac{1}{s+\beta}\Big)\,ds = \lim_{n\to\infty}\Big[\log\frac{s+\alpha}{s+\beta}\Big]_{s=0}^{s=N} = \log\frac{\beta}{\alpha}\,. \end{aligned}$$

Problem 3.3.5 Calculate the integral $f(t) = \displaystyle\int_0^\infty \frac{\cos xt}{1+x^2}\,dx\,.$

Hint: Calculate $F(z) = \mathcal{L}[f](z)$ using Fubini's theorem. Then, calculate the inverse Laplace transform of $F(z)$.
Solution: Applying Fubini's theorem, we have that:

$$\begin{aligned} L[f](z) &= \int_0^\infty \Big(\int_0^\infty \frac{\cos xt}{1+x^2}\,dx\Big) e^{-zt}\,dt = \int_0^\infty \Big(\int_0^\infty e^{-zt}\cos xt\,dt\Big)\frac{dx}{1+x^2} \\ &= \int_0^\infty \frac{L[\cos xt](z)}{1+x^2}\,dx = \int_0^\infty \frac{z}{(z^2+x^2)(1+x^2)}\,dx\,. \end{aligned}$$

Now, decomposing into simple fractions, we obtain that:

$$\frac{1}{(z^2+x^2)(1+x^2)} = \frac{Ax+B}{z^2+x^2}+\frac{Cx+D}{1+x^2}\,, \qquad \text{with } A=C=0\,,\quad B=-D=-\frac{1}{z^2-1}\,.$$

Hence,

$$\begin{aligned} L[f](z) &= \frac{-z}{z^2-1}\int_0^{\infty}\frac{dx}{z^2+x^2} + \frac{z}{z^2-1}\int_0^{\infty}\frac{dx}{1+x^2} \\ &= \frac{z}{z^2-1}\lim_{N\to\infty}\big[\arctan x\big]_{x=0}^{x=N} - \frac{1}{z^2-1}\lim_{N\to\infty}\Big[\arctan\frac{x}{z}\Big]_{x=0}^{x=N} \\ &= \frac{\pi}{2}\left(\frac{z}{z^2-1} - \frac{1}{z^2-1}\right) = \frac{\pi}{2}\,\frac{1}{z+1}\,, \end{aligned}$$

and from this we obtain that $f(x) = \frac{\pi}{2}\,e^{-x}$, $(x \geq 0)$. Finally, as the cosine function is even, $f(x)$ also is even, and then:

$$f(x) = \frac{\pi}{2}\,e^{-|x|}\,, \qquad x \in \mathbb{R}\,.$$

Problem 3.3.6 Solve the following initial value problems, obtaining first the Laplace transform $Y(z)$ of the solution $y(t)$ and then, anti-transforming $Y(z)$:

$$1)\ \begin{cases} y' - 5y = \cos 3t\,, \\ y(0) = 1/2\,, \end{cases} \qquad 2)\ \begin{cases} y'' + 16y = \cos 4t\,, \\ y(0) = 0\,,\ y'(0) = 1\,, \end{cases}$$

$$3)\ \begin{cases} y'' - 6y' + 9y = t^2e^{3t}\,, \\ y(0) = 2\,,\ y'(0) = 6\,, \end{cases} \qquad 4)\ \begin{cases} y'' + 4y' + 6y = 1 + e^{-t}\,, \\ y(0) = 0\,,\ y'(0) = 0\,. \end{cases}$$

Solution: 1) Transforming the equation, using the property (5) of Laplace transforms and denoting $Y(z) = L[y(t)](z)$, we have:

$$zY(z) - y(0) - 5Y(z) = L[\cos 3t](z) \qquad \Longrightarrow \qquad (z-5)Y(z) - 1/2 = \frac{z}{z^2+9}\,.$$

Hence,

$$Y(z) = \frac{1}{2}\,\frac{1}{z-5} + \frac{z}{(z-5)(z^2+9)}\,.$$

Decomposing into simple fractions we get

$$\frac{z}{(z-5)(z^2+9)} = \frac{A}{z-5} + \frac{Bz+C}{z^2+9}\,, \qquad \text{with } A = -B = \frac{5}{34}\,,\ C = \frac{9}{34}\,.$$

Therefore, from the table of Laplace transforms, we get

$$\begin{aligned} y(t) &= \frac{1}{2}\,L^{-1}\Big[\frac{1}{z-5}\Big](t) + \frac{5}{34}\,L^{-1}\Big[\frac{1}{z-5}\Big](t) - \frac{5}{34}\,L^{-1}\Big[\frac{z}{z^2+9}\Big](t) + \frac{9}{34}\,L^{-1}\Big[\frac{1}{z^2+9}\Big](t) \\ &= \frac{1}{2}\,e^{5t} + \frac{5}{34}\,e^{5t} - \frac{5}{34}\cos 3t + \frac{9}{34}\,\frac{1}{3}\sin 3t = \frac{11}{17}\,e^{5t} - \frac{5}{34}\cos 3t + \frac{3}{34}\sin 3t\,. \end{aligned}$$

2) Transforming the equation, using the property (5) of Laplace transforms and denoting $Y(z) = L[y(t)](z)$, we have:

$$z^2Y(z) - zy(0) - y'(0) + 16Y(z) = L[\cos 4t](z) \qquad \Longrightarrow \qquad (z^2+16)Y(z) - 1 = \frac{z}{z^2+16}\,,$$

and solving for $Y(z)$ we obtain that:

$$Y(z) = \frac{1}{z^2+16} + \frac{z}{(z^2+16)^2}\,.$$

Hence, using the table of Laplace transforms and the part 4) of problem 3.3.3, we get:

$$y(t) = \frac{1}{4}\sin 4t + \frac{1}{8}\,t\,\sin 4t\,.$$

3) Transforming the equation, using the property (5) of Laplace transforms and denoting $Y(z) = L[y(t)](z)$, we have:

$$z^2Y(z) - zy(0) - y'(0) - 6zY(z) + 6y(0) + 9Y(z) = L[t^2e^{3t}](z)\,,$$

that is to say,

$$(z^2 - 6z + 9)Y(z) - 2z + 6 = \frac{2}{(z-3)^3}\,,$$

and so

$$Y(s) = \frac{2}{z-3} + \frac{2}{(z-3)^5} \qquad \Longrightarrow \qquad y(t) = 2e^{3t} + \frac{1}{12}\,t^4e^{3t}\,.$$

4) Transforming the equation, using the property (5) of Laplace transforms and denoting $Y(z) = L[y(t)](z)$, we have:

$$z^2Y(z) - zy(0) - y'(0) + 4zY(z) - 4y(0) + 6Y(z) = L[1](z) + L[e^{-t}](z)\,,$$

that is to say,

$$(z^2 + 4z + 6)Y(z) = \frac{1}{z} + \frac{1}{z+1} \qquad \Longrightarrow \qquad Y(z) = \frac{2z+1}{z(z+1)((z+2)^2+2)}\,.$$

Decomposing into simple fractions, we obtain that

$$Y(z) = \frac{1}{6}\frac{1}{z} + \frac{1}{3}\frac{1}{z+1} + \frac{(-1/2)z - 5/3}{(z+2)^2+2} = \frac{1}{6}\frac{1}{z} + \frac{1}{3}\frac{1}{z+1} - \frac{1}{2}\frac{z+2}{(z+2)^2+2} - \frac{2}{3}\frac{1}{(z+2)^2+2}\,.$$

Hence, using the table of Laplace transforms, we deduce that

$$y(t) = \frac{1}{6} + \frac{1}{3}\,e^{-t} - \frac{1}{2}\,e^{-2t}\cos\sqrt{2}\,t - \frac{\sqrt{2}}{3}\,e^{-2t}\sin\sqrt{2}\,t\,.$$

Problem 3.3.7 Solve the initial value problem for the system of differential equations:

$$\begin{cases} x' - 6x + 3y = 8e^t\,, \\ y' - 2x - y = 4e^t\,, \\ x(0) = -1\,,\ y(0) = 0\,. \end{cases}$$

Solution: Transforming the equation, using the property (5) of Laplace transforms and denoting $X(s) = L[x(t)](s)$, $Y(s) = L[y(t)](s)$, we obtain the system of algebraic equations:

$$\begin{cases} zX(z) - x(0) - 6X(z) + 3Y(z) = 8L[e^t](z)\,, \\ zY(z) - y(0) - 2X(z) - Y(z) = 4L[e^t](z)\,, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} (z-6)X(z) + 3Y(z) = -1 + \frac{8}{z-1}\,, \\ -2X(z) + (z-1)Y(z) = \frac{4}{z-1}\,. \end{cases}$$

Multiplying the first equation by 2 and the second one by $z-6$ and then summing the two new equations, we deduce that

$$\begin{cases} 2(z-6)X(z) + 6Y(z) = \frac{18-2z}{z-1}\,, \\ -2(z-6)X(s) + (z-1)(z-6)Y(z) = \frac{4z-24}{z-1}\,, \end{cases} \qquad \Longrightarrow \qquad (z^2 - 7z + 12)Y(z) = \frac{2z-6}{z-1}\,,$$

and so,

$$Y(z) = \frac{2z-6}{(z-1)(z-3)(z-4)} = \frac{2}{(z-1)(z-4)}\,.$$

Decomposing into simple fractions

$$Y(z) = \frac{A}{z-1} + \frac{B}{z-4}\,, \qquad \text{with } B = -A = \frac{2}{3}\,.$$

Hence,

$$y(t) = -\frac{2}{3}\,e^{t} + \frac{2}{3}\,e^{4t} = \frac{2}{3}\,(e^{4t} - e^{t})\,.$$

To obtain $x(t)$ it is not necessary to compute $X(s)$ and then to anti-transform, since, from the second differential equation of the system we obtain that

$$x(t) = \frac{1}{2}\,(y'(t) - y(t) - 4e^{t}) = e^{4t} - 2e^{t}\,.$$

Problem 3.3.8 Solve the following initial value problems:

$$1)\ \begin{cases} y'' + y' = \begin{cases} t+1 & \text{if } \ 0 < t < 1\,, \\ 3-t & \text{if } \ t > 1\,, \end{cases} \\ y(0) = -1,\ y'(0) = 0\,. \end{cases} \qquad 2)\ \begin{cases} y'' + 4y = \begin{cases} \cos 2t & \text{if } \ 0 < t < 2\pi\,, \\ 0 & \text{if } \ t > 2\pi\,, \end{cases} \\ y(0) = y'(0) = 0\,. \end{cases}$$

Solution: 1) Let us express the function in the second member of the equation in terms of the Heaviside function:

$$g(t) = \begin{cases} t+1 & \text{if } 0 < t < 1\,, \\ 3-t & \text{if } t > 1\,, \end{cases} = t + 1 - 2(t-1)H(t-1)\,.$$

Hence, using the property (4) of the Laplace transform:

$$L[g(t)](z) = L[t+1](z) - 2\,L[(t-1)H(t-1)](z) = \frac{1}{z^2} + \frac{1}{z} - 2e^{-z}L[t](z) = \frac{1}{z^2} + \frac{1}{z} - 2e^{-z}\frac{1}{z^2}\,.$$

Therefore, applying the Laplace transform to the differential equation, we obtain that

$$L[y''](z) + L[y'](z) = L[g](z) \implies z^2Y(z) - zy(0) - y'(0) + zY(z) - y(0) = \frac{z+1}{z^2} - 2e^{-z}\frac{1}{z^2}\,,$$

and so, substituting the initial values, we arrive at

$$Y(z) = \frac{1}{z^3} - \frac{1}{z} - 2\frac{e^{-z}}{z^3(z+1)}\,.$$

But, decomposing into simple fractions we have that

$$\frac{1}{z^3(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z+1}\,, \qquad \text{with } A = -B = C = -D = 1\,,$$

and so

$$L^{-1}\Big[\frac{1}{z^3(z+1)}\Big](t) = 1 - t + \frac{t^2}{2} - e^{-t}\,.$$

Hence, using again the property (4) of the Laplace transform, we obtain finally that:

$$\begin{aligned} y(t) &= \frac{t^2}{2} - 1 - 2\Big(L^{-1}\Big[\frac{1}{z^3(z+1)}\Big](t-1)\Big)\,H(t-1) \\ &= \frac{t^2}{2} - 1 - 2\Big(1-(t-1)+\frac{(t-1)^2}{2} - e^{-(t-1)}\Big)\,H(t-1) \\ &= \frac{t^2}{2} - 1 - \big(4-2t+(t-1)^2-2e^{-(t-1)}\big)\,H(t-1) \\ &= \frac{t^2}{2} - 1 - \big(t^2-4t+5-2e^{1-t}\big)\,H(t-1)\,. \end{aligned}$$

2) Let us express the function in the second member of the equation in terms of the Heaviside function:

$$g(t) = \begin{cases} \cos(2t-4\pi)\,, & \text{if } 0<t<2\pi\,, \\ 0\,, & \text{if } t>2\pi\,, \end{cases} = \cos 2(t-2\pi) - \cos 2(t-2\pi)\,H(t-2\pi)\,,$$

and so its Laplace transform is

$$\begin{aligned} L[g](t) &= L[\cos 2(t-2\pi)](z) - L[\cos 2(t-2\pi)\,H(t-2\pi)](z) \\ &= \frac{z}{z^2+4} - e^{-2\pi z} L[\cos 2t](z) = \frac{z(1-e^{-2\pi z})}{z^2+4}\,. \end{aligned}$$

Hence, applying the Laplace transform to the differential equation, we obtain that

$$L[y''](z) + 4\,L[y](z) = L[g](z) \quad \Longrightarrow \quad z^2Y(s) - zy(0) - y'(0) + 4Y(z) = \frac{z(1-e^{-2\pi z})}{z^2+4}\,,$$

and so, substituting the initial values and solving for $Y(z)$, we arrive at

$$Y(z) = \frac{z(1-e^{2\pi z})}{(z^2+4)^2}\,.$$

Therefore,

$$\begin{aligned} y(t) &= L^{-1}\Big[\frac{z}{(z^2+4)^2}\Big](t) - L^{-1}\Big[e^{-2\pi z}\frac{z}{(z^2+4)^2}\Big](t) \\ &= L^{-1}\Big[\frac{z}{(z^2+4)^2}\Big](t) - L^{-1}\Big[\frac{z}{(z^2+4)^2}\Big](t-2\pi)H(t-2\pi)\,, \end{aligned}$$

where we have used the property (4) of Laplace transform. Using the part 4) of problem 3.3.3 finally we get

$$y(t) = \frac{1}{4}\,t\sin 2t - \frac{1}{4}\,(t-2\pi)\sin 2t\,H(t-2\pi)\,.$$

Problem 3.3.9 Let us consider the differential equation $tx'' + 2x' + tx = 0$ for $t > 0$.

a) Find the differential equation that verifies the Laplace transform $X(z)$ of $x(t)$ with the initial data $x(0) = 1$.

b) Solve the differential equation for $X(z)$ using the property $\lim_{z\to+\infty} X(z) = 0$.

c) Calculate the Laplace anti-transform $x(t)$.

Solution: a) We have that

$$L[tx''(t)](z) + 2L[x'(t)](z) + L[tx(t)](z) = 0\,,$$

and using the properties (5) and (7) of the Laplace transform, we obtain that

$$-\frac{d}{dz}\big(L[x''(t)](z)\big) + 2\big(zL[x(t)](z) - x(0)\big) - \frac{d}{dz}\big(L[x(t)](z)\big) = 0$$

and using the property (6):

$$-\frac{d}{dz}\big(z^2X(z) - zx(0) - x'(0)\big) + 2zX(z) - 2 - X'(z) = 0\,.$$

Simplifying, since $x'(0)$ is a constant,

$$-\frac{d}{dz}\big(z^2X(z) - z\big) + 2zX(z) - 2 - X'(z) = 0\,,$$

that is to say,

$$(z^2+1)X'(z) + 1 = 0\,.$$

Let us observe that, from the differential equation, we immediately deduce that $x'(0) = 0$, but this fact is irrelevant, since the derivative of such constant is zero independently of its value.
b) Hence, $X'(z) = -1/(z^2+1)$, and so $X(z) = C - \arctan z$. But, from Riemann-Lebesgue theorem for the Laplace transform, we know that $\lim_{z\to+\infty} X(z) = 0$, and so we obtain that necessarily $C = \pi/2$. Therefore, $X(z) = \pi/2 - \arctan z = \arctan(1/z)$.
c) By the problem 3.3.1 we deduce that $x(t) = \dfrac{\sin t}{t}$.

Problem 3.3.10 Solve the following initial value problems:

$$1)\ \begin{cases} y'' = \delta_a\,, \\ y(0) = 0\,,\ y'(0) = 0\,. \end{cases} \qquad\qquad 2)\ \begin{cases} y' + 8y = \delta_1 + \delta_2\,, \\ y(0) = 0\,. \end{cases}$$

Solution: 1) Applying the Laplace transform to the equation and using the table of transforms, we have that:

$$z^2Y(z) - zy(0) - y'(0) = L[\delta(t-a)](z) \quad\Longrightarrow\quad z^2Y(z) = e^{-az}\,,$$

and so

$$Y(z) = e^{-az}\frac{1}{z^2} = e^{-az}L[t](z)$$

and by the property (4) of the Laplace transform we finally obtain that

$$y(t) = (t-a)\,H(t-a)\,.$$

b) Applying the Laplace transform to the equation and using the table of transforms, we have that:

$$zY(z) - y(0) + 8Y(z) = L[\delta(t-1)](z) + L[\delta(t-2)](z) \quad\Longrightarrow\quad (z+8)Y(z) = e^{-z} + e^{-2z}\,,$$

and so

$$Y(z) = (e^{-z} + e^{-2z})\,\frac{1}{z+8} = (e^{-z} + e^{-2z})\,L[e^{-8t}](z)$$

and by the property (4) of the Laplace transform we finally obtain that

$$y(t) = e^{-8(t-1)}\, H(t-1) + e^{-8(t-2)}\, H(t-2)\,.$$

Problem 3.3.11 a) Let us consider the Volterra integral equation

$$y(t) + \int_0^t k(t-x)\, y(x)\, dx = f(t)\,,$$

where f and k are known functions and $y(t)$ is the unknown function. Calculate the Laplace transform of $y(t)$ in terms of the transforms of k and f.
b) As an application, solve the Volterra equation

$$y(t) - 2\int_0^t \cos(t-x)\, y(x)\, dx = e^{2t}\,.$$

Solution: Let us observe that the equation can be written as $y + k * y = f$ where $*$ denotes the convolution. Hence, applying the Laplace transform to the equation and using Proposition 3.7, we obtain that

$$Y(z) + L[k](z)Y(z) = L[f](z) \qquad \Longrightarrow \qquad Y(z) = \frac{L[f](z)}{1 + L[k](z)}\,.$$

In particular, in the case of the indicated Volterra equation we obtain

$$Y(z) = \frac{1}{z-2}\,\frac{1}{1 - 2z/(z^2+1)} \qquad \Longrightarrow \qquad Y(z) = \frac{z^2+1}{(z-1)^2(z-2)}\,.$$

Decomposing into simple fractions we get

$$Y(z) = \frac{5}{z-2} - \frac{4}{z-1} - \frac{2}{(z-1)^2}\,,$$

and, using the table of Laplace transforms, we finally obtain

$$y(t) = 5e^{2t} - (4+2t)\, e^t\,.$$

Problem 3.3.12 Solve, for $\omega \neq \omega_0$, the initial value problem

$$\begin{cases} x'' + \omega_0^2\, x = k \sin \omega t\,, & \text{if } t > 0\,, \\ x(0) = x'(0) = 0\,, \end{cases}$$

which describes the forced oscillations of a mass in a not damped spring. What happens if $\omega = \omega_0$? Explain physically the obtained results.

Solution: Applying the Laplace transform to the equation we obtain:

$$L[x''](z) + \omega_0^2\, L[x](z) = k\, L[\sin \omega t](z) \implies z^2 L[x](z) - z\, x(0) - x'(0) + \omega_0^2\, L[x](z) = \frac{k\omega}{z^2+\omega^2}$$

and therefore,

$$X(z) = L[x](z) = \frac{k\omega}{(z^2+\omega^2)(z^2+\omega_0^2)}\,.$$

Decomposing this fraction into simple fractions we obtain:

$$L[x](z) = \frac{k\omega}{(z^2+\omega^2)(z^2+\omega_0^2)} = \frac{Az+B}{z^2+\omega^2} + \frac{Cz+D}{z^2+\omega_0^2}$$

with

$$A = C = 0 \qquad \text{and} \qquad B = -D = \frac{k\omega}{\omega_0^2-\omega^2}\,.$$

Therefore:

$$L[x](z) = \frac{B}{z^2+\omega^2} + \frac{D}{z^2+\omega_0^2} \implies x(t) = \frac{B}{\omega}\sin\omega t + \frac{D}{\omega_0}\sin\omega_0 t = \frac{k}{\omega_0}\,\frac{\omega_0\sin\omega t - \omega\sin\omega_0 t}{\omega_0^2-\omega^2}\,.$$

If $\omega = \omega_0$, then

$$X(z) = \frac{k\omega_0}{(z^2+\omega_0^2)^2}$$

and using the part 3) of problem 3.3.3, we obtain that

$$x(t) = \frac{k}{2\omega_0^2}\sin\omega_0 t - \frac{k}{2\omega_0}\,t\cos\omega_0 t\,.$$

In the first case, we obtain a periodic response of the spring to the periodic forcing $k\sin\omega t$. In the second case the response is not periodic and the reason is that the natural frequency ω_0 of the spring coincides with the forcing frequency ω. This phenomenon is known as *resonance.*

LAPLACE TRANSFORMS TABLE

$$f(t) = 1\,, \qquad \mathcal{L}[f](z) = \frac{1}{z} \qquad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^n \quad (n \in \mathbb{N})\,, \qquad \mathcal{L}[f](z) = \frac{n!}{z^{n+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^a \quad (a > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(a+1)}{z^{a+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2+a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2+a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \frac{\sin at}{t} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \arctan\frac{a}{z} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = e^{at} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{1}{z-a} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}t^b \quad (a \in \mathbb{R},\, b > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(b+1)}{(z-a)^{b+1}} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}\sin bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{b}{(z-a)^2+b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}\cos bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z-a}{(z-a)^2+b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = \sinh at = \frac{e^{at}-e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2-a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \cosh at = \frac{e^{at}+e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2-a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \delta_a \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = e^{-az}\,,$$

$$f(t) = \sin at - at\cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2a^3}{(z^2+a^2)^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t\sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2az}{(z^2+a^2)^2} \quad (\operatorname{Re} z > 0)\,.$$