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Departamento de Matemáticas

# Integration and Measure. Problems

## Chapter 1: Measure theory

### Section 1.1: Measurable spaces

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# 1 Measure Theory

## 1.1 Measurable spaces

**Problem 1.1.1** Let $f : X \longrightarrow Y$ be a mapping. Given a subset $A \subseteq Y$ let us define:

$$f^{-1}(A) = \{x \in X : \ f(x) \in A\}\,.$$

Prove that

*i)* $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

*ii)* $f^{-1}(\bigcup_j A_j) = \bigcup_j f^{-1}(A_j)$.

*iii)* $f^{-1}(\bigcap_j A_j) = \bigcap_j f^{-1}(A_j)$.

*Hint:* To prove that two sets $A$ and $B$ are equal you must prove that each element belonging to $A$ also belongs to $B$ and reciprocally each element in $B$ also belongs to $A$.

**Problem 1.1.2** Let $f : X \longrightarrow Y$ be a mapping between two topological spaces $(X, \mathcal{T})$, $(Y, \mathcal{T}')$. Prove that $f$ is continuous if and only if $f$ is continuous at every $x \in X$.

*Hint:* To prove an statement of type $A \iff B$ you must prove that if we assume that $A$ holds, then $B$ also holds and viceversa.

**Problem 1.1.3**

*i)* Show that if $X = \{1, 2, 3\}$, then $\mathcal{F} := \{\emptyset, \{2, 3\}, X\}$ is not a $\sigma$-algebra.

*ii)* Let $X = \{a, b, c, d\}$. Check that the family of subsets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$$

is a $\sigma$-algebra in $X$.

*Hint:* You must check if the properties of a $\sigma$-algebra are satisfied.

**Problem 1.1.4** Let $\mathcal{S}$ be a family of subsets of $X$, $\mathcal{S} \subseteq \mathcal{P}(X)$. Prove that

$$\mathcal{A}_{\mathcal{S}} = \bigcap \{\mathcal{A} : \ \mathcal{A} \text{ is a } \sigma\text{-algebra}, \ \mathcal{S} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)\}$$

is the smallest $\sigma$-algebra containing $\mathcal{S}$.

*Note:* $\mathcal{A}_{\mathcal{S}}$ is called the $\sigma$-algebra generated by $\mathcal{S}$ and sometimes is denoted as $\sigma(\mathcal{S})$.
*Hint:* Prove that $\mathcal{A}_{\mathcal{S}}$ is a $\sigma$-algebra.

**Problem 1.1.5** Let $X = \{a, b, c, d\}$. Construct the $\sigma$-algebra generated by

$$\mathcal{E}_1 = \{\{a\}\} \ \text{ y } \ \text{por} \ \ \mathcal{E}_2 = \{\{a\}, \{b\}\}\,.$$

*Hint:* To construct them you must add the necessary subsets so that the $\sigma$-algebra properties are verified.

**Problem 1.1.6** Show with an example that the union of two $\sigma$-algebras does not have to be a $\sigma$-álgebra.

*Hint:* It suffices to consider a three-point set $X$.

**Problem 1.1.7** Determine the $\sigma$-algebra generated by the collection of all finite subsets of a non-countable set $X$.

**Problem 1.1.8** Consider the $\sigma$-algebra of borelian subsets in $\mathbb{R}$. Is the following true or false?: There is a subset $A$ of $\mathbb{R}$ which is not measurable, but such that $B = \{x \in A : x \text{ is irrational}\}$ is measurable.

*Hint:* Consider the set $C = \{x \in A : x \text{ is rational}\}$.

**Problem 1.1.9** Let $(X, \mathcal{A})$ be a measurable space and $(Y, \mathcal{T})$ be a topological space. Let us consider a mapping $f : X \longrightarrow Y$. Prove that

*i)* The collection $\mathcal{A}' = \{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$ is a $\sigma$-algebra in $Y$. $\mathcal{A}'$ is called the *image $\sigma$-algebra of* $\mathcal{A}$.

*ii)* If $f$ is measurable, then $\mathcal{B}(Y) \subseteq \mathcal{A}'$. Equivalently, if $E$ is a borel set in $Y$, then $f^{-1}(E) \in \mathcal{A}$ and so $E \in \mathcal{A}'$.

*Hint:* ii) Prove that $\mathcal{T} \subseteq \mathcal{A}'$.

**Problem 1.1.10** Let $g : X \to Y$ be a mapping. Let $\mathcal{A}$ be a $\sigma$-algebra in $Y$. Prove that $\mathcal{A}' = \{g^{-1}(E) : E \in \mathcal{A}\}$ is a $\sigma$-algebra in $X$. $\mathcal{A}'$ is called the *pre-image $\sigma$-algebra of* $\mathcal{A}$.

**Problem 1.1.11** A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *algebra* if the following conditions hold:

(1) $\emptyset \in \mathcal{A}$,

(2) $A \in \mathcal{A} \implies A^c \in \mathcal{A}$,

(3) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$.

Prove that an algebra $\mathcal{A}$ in $X$ is a $\sigma$-álgebra if and only if it is closed for increasing countable unions of sets, that is to say:

$$E_i \in \mathcal{A}, \quad E_1 \subset E_2 \subset \dots \qquad \Longrightarrow \qquad \bigcup_1^\infty E_i \in \mathcal{A}\,.$$

**Problem 1.1.12** Let $u, v : X \longrightarrow \mathbb{R}$ be measurable functions and let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Prove that

*i)* $\varphi \circ u$ is measurable.

*ii)* $u + v$, $uv$, $|u|^\alpha$ $(\alpha > 0)$ are measurable functions.

*iii)* If $u(x) \neq 0$ for all $x \in X$, then $1/u$ is measurable.

*iv)* If $f = u + iv$, then $f : X \longrightarrow \mathbb{C}$ is measurable.

*v)* The previous exercises *i) ii) iii)* are also valid for $u, v : X \longrightarrow \mathbb{C}$ measurable functions and $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ continuous.

*vi)* If $u, v : X \longrightarrow \mathbb{R}$ and $f = u + iv$ is measurable, then $u$, $v$ and $|f|$ are real measurable.

**Problem 1.1.13** Let $(X, \mathcal{A})$ be a measurable space and let $f : X \longrightarrow \mathbb{R}$ be a function. Prove that the following assertions are equivalent:

*i)* $\{x \in X : f(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

*ii)* $\{x \in X : f(x) \geq \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

*iii)* $\{x \in X : f(x) < \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

*iv)* $\{x \in X : f(x) \leq \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

*v)* $f^{-1}(I) \in \mathcal{A}$ for every interval $I$.

*vi)* $f$ is measurable, that is to say that $f^{-1}(V) \in \mathcal{A}$ for every open set $V$.

*vii)* $f^{-1}(F) \in \mathcal{A}$ for every closed set $F$.

*viii)* $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B$.

*Hint:* $\mathcal{A}' = \{E \subseteq \mathbb{R} : \ f^{-1}(E) \in \mathcal{A}\}$ is a $\sigma$-algebra in $\mathbb{R}$ (in fact it is the image $\sigma$-algebra of $\mathcal{A}$) and $\mathcal{B}(\mathbb{R}) = \sigma(\{(\alpha, \infty) : \alpha \in \mathbb{R}\})$.

**Problem 1.1.14** Prove that the previous problem is also valid if $f : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$. Recall that by interval, open set, closed set or Borel set in $\overline{\mathbb{R}}$ we understand the corresponding concept in $\mathbb{R}$ joining it $-\infty$, $+\infty$ or both or neither.

**Problem 1.1.15** Prove that if $f$ is a real function on a measurable space $X$ such that $\{x \in X : \ f(x) \geq r\}$ is measurable for every rational $r$, then $f$ is measurable.

*Hint:* Given any $\alpha \in \mathbb{R}$ there exists a sequence $\{r_n\}$ of rational numbers such that $r_n \nearrow \alpha$ as $n \to \infty$. Use problem 1.1.13.

**Problem 1.1.16** Let $\mathcal{M}$ be the $\sigma$-algebra in $\mathbb{R}$ given by $\mathcal{M} = \{\emptyset, (-\infty, 0], (0, \infty), \mathbb{R}\}$. Let $g$ be the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty, 0], \\ 1 & \text{if} \quad x \in (0, 1], \\ 2 & \text{if} \quad x \in (1, \infty). \end{cases}$$

Is $g$ measurable? How are the measurable functions $f : (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

**Problem 1.1.17**

a) Prove that if $f : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \longrightarrow \mathbb{R}$ is a continuous function, then $f$ is measurable.

b) Prove that if $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow \mathbb{R}$ is an increasing function, then $f$ is measurable.

c) Let $(X, \mathcal{A})$ be a measurable space. Given $A \subset X$, let $\chi_A$ be the characteristic function of $A$. Prove that $\chi_A$ is measurable if and only if $A$ is measurable.

*Hints:* b) What can you say about $f^{-1}(I)$ when $I$ is an interval? c) Who are $\chi_A^{-1}(0)$ and $\chi_A^{-1}(1)$?

**Problem 1.1.18** Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\overline{\mathbb{R}} = [\infty, \infty]$. Prove that

a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$ .
b) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ .
c) If $a_n \le b_n$ for all $n$, then $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$ .
d) Show with an example that strict inequality can hold in part b).

*Hint:* d) Consider the sequences $a_n = (-1)^n$, $b_n = (-1)^{n+1}$.

**Problem 1.1.19**

a) Prove that if $f, g : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ are measurable functions, then $\max\{f, g\}$ and $\min\{f, g\}$ are also measurable functions.
b) Prove that if $f_n : X \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ is a sequence of measurable functions, then

$$\sup_n f_n \ , \qquad \inf_n f_n \ , \qquad \limsup_{n\to\infty} f_n \ , \qquad \liminf_{n\to\infty} f_n$$

are measurable functions.
c) Prove that the limit of every pointwise convergent sequence of measurable functions is measurable.

*Hint:* b) If $g = \sup_k f_k$ then $\{x : g(x) > \alpha\} = \cup_k \{x : f_k(x) > \alpha\}$.

**Problem 1.1.20** Suppose that $f, g : X \longrightarrow \mathbb{R}$ are measurable. Prove that the sets

$$\{x \in X : \ f(x) < g(x)\}, \qquad \{x \in X : \ f(x) = g(x)\}$$

are measurable.

**Problem 1.1.21** Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

*Hint:* The set $A$ of points at which $\{f_n\}$ converges to a finite limit verifies $A = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{i,j\ge m} \{x : |f_i(x) - f_j(x)| < \frac{1}{n}\}$.