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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 1: Measure theory

Section 1.2: Measure spaces

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1 Measure Theory

1.2. Measure spaces

Problem 1.2.1 Let X be a set and $\mathcal{A} = \mathcal{P}(X)$. Let us also consider a function $p : X \longrightarrow [0, \infty]$. Now, we define for $A \subseteq X$ the set function

$$\mu(A) := \sum_{x \in A} p(x) = \sup_{\{x_1,\dots,x_n\} \subseteq A} \sum_{j=1}^{n} p(x_j)\,.$$

Prove that μ is a measure on X. In the particular case that $p(x) = 1$ for all $x \in X$, this measure is known as the *counting measure* in X, since in this case $\mu(A) = \sum_{x \in A} 1 = \#A$, the number of elements of A.

Problem 1.2.2 Let $(X, \mathcal{A})$ be a measurable space and define the function $\delta_{x_0} : \mathcal{A} \longrightarrow [0, \infty]$ by

$$\delta_{x_0}(A) = \begin{cases} 1\,, & \text{if } x_0 \in A\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Prove that δ_{x_0} is a measure on $(X, \mathcal{A})$ (it is called the δ-Dirac measure concentrated at x_0).

Problem 1.2.3 Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \longrightarrow [0, \infty]$ be a countably additive function on the σ-algebra $\mathcal{A}$.

a) Show that if μ satisfies that $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\varnothing) = 0$ (and therefore μ is a measure).
b) Find an example for which $\mu(\varnothing) \neq 0$ (and therefore the countably subadditivity property does not imply that μ is a measure).

Hint: b) Take $\mu(A) = \infty$ for any set A.

Problem 1.2.4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)\,.$$

Problem 1.2.5 Let $(X, \mathcal{M}, \mu)$ be a measure space. Given $E \in \mathcal{M}$ we define

$$\mu_E(A) = \mu(A \cap E)\,, \qquad \text{for all } A \in X\,.$$

Prove that que μ_E is also a measure on $(X, \mathcal{M})$. We say that μ_E is *concentrated* at E because $\mu_E(A) = 0$ when $A \subseteq E^c$.

Problem 1.2.6 Let X be an infinite countable set. Let us consider the σ-algebra $\mathcal{M} = \mathcal{P}(X)$ and let us define for $A \in \mathcal{M}$:

$$\mu(A) = \begin{cases} 0\,, & \text{if } A \text{ is finite,} \\ \infty\,, & \text{if } A \text{ is infinite.} \end{cases}$$

a) Prove that μ is finitely additive, but not countably additive.
b) Prove that $X = \lim_{n\to\infty} A_n$, being $\{A_n\}_{n=1}^{\infty}$ an increasing sequence of sets such that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$.

Problem 1.2.7 Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and μ be the counting measure on X. Construct a decreasing sequence of subsets $A_n \in \mathcal{P}(\mathbb{N})$ such that $\cap_n A_n = \varnothing$, but $\lim_{n\to\infty} \mu(A_n) \neq 0$.

Problem 1.2.8* Let $(X, \mathcal{A})$ be a measurable space. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on $(X, \mathcal{A})$.

a) Prove that if $\{\mu_n\}_{n=1}^{\infty}$ is increasing, that is to say that

$$\mu_n(A) \leq \mu_{n+1}(A)\,, \qquad \forall A \in \mathcal{A},$$

then

$$\mu(A) := \lim_{n\to\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

b) Prove that for any sequence of measures $\{\mu_n\}_{n=1}^{\infty}$

$$\mu(A) = \sum_{n=1}^{\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

Hints: a) Consider a countable disjoint family $\{A_j\} \subset \mathcal{A}$ and let $A = \cup_j A_j$. If $\mu(A) = \infty$, then for all $M \in \mathbb{N}$, $\exists N = N(M)$ such that $\mu_n(A) > M$ for all $n \geq N$. Prove that then $\exists K \in \mathbb{N}$ such that $\sum_{j=1}^{K} \mu(A_j) > M - 1$. If $\mu(A) < \infty$, then $\mu_n(A) = \sum_{j=1}^{\infty} \mu_n(A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$ and so, $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$. Also, $\mu_n(A) = \sum_{j=1}^{\infty} \mu_n(A_j) \geq \sum_{j=1}^{K} \mu_n(A_j)$ and so, $\mu(A) \geq \sum_{j=1}^{K} \mu(A_j)$ for every K. Hence, $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$. b) Take $\nu_n = \sum_{j=1}^{n} \mu_j$ and apply a).

Problem 1.2.9 Let $(X, \mathcal{M}, \mu)$ be a measure space such that for all $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$. A measure space or a measure with this property is called *semifinite*.

a) Show that a σ-finite measure is semifinite.

b) Let X be a non countable set. Let $\mathcal{M} = \mathcal{P}(X)$. Let μ be the counting measure. Prove that μ is semifinite but it is not σ-finite.

Problem 1.2.10 Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space and let $E \in \mathcal{M}$ be a set with $\mu(E) = \infty$.

a) Prove that

$$\sup\{\mu(F) : \; F \in \mathcal{M}, F \subset E, \mu(F) < \infty\} = \infty\,.$$

b) Prove that if c is a positive real number, then there exists a set $F \subset E$ such that $F \in \mathcal{M}$ and $c < \mu(F) < \infty$.

Hint: a) Denote by s the supremum and suppose that $s < \infty$. Show that there exists $F \subset E$ with $\mu(F) = s$. But then if $E' = E \setminus F$ then $\mu(E') = \infty$ and $\exists F' \subset E'$ with $0 < \mu(E') < \infty$. Get a contradiction with the set $F \cup F'$.

Problem 1.2.11 Let $\{A_n\}$ be measurable sets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that x belongs to only a finite number of A_n's for a.e. $x \in X$. Alternatively, the set A of points x belonging to an infinite number of A_n's, has zero measure (Borel-Cantelli Lemma).

Hint: $A = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$.

Problem 1.2.12* Let $(X, \mathcal{A}, \mu)$ be a measure space, and let

$$\mathcal{N} = \{N \subseteq X : \ N \subseteq B \in \mathcal{A},\ \mu(B) = 0\}.$$

Prove that

i) $\overline{\mathcal{A}} = \{A \cup N : \ A \in \mathcal{A},\ N \in \mathcal{N}\}$ is a σ-algebra. In fact, $\overline{\mathcal{A}}$ is the σ-algebra generated by $\mathcal{A} \cup \mathcal{N}$.

ii) $\overline{\mu} : \overline{\mathcal{A}} \longrightarrow [0, \infty]$ given by $\overline{\mu}(A \cup N) = \mu(A)$ is a well-defined measure and extends μ.

iii) $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space.