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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.3: Integration on product spaces

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2 Integration Theory

2.3. Integration on product spaces

Problem 2.3.1 Prove that $f(x) = e^{-x^2} \in L^1(\mathbb{R})$ and calculate $I = \displaystyle\int_{\mathbb{R}} e^{-x^2}\, dx$.

Hint: $x^2 \geq x$ for $x \geq 1$. Relate I^2 with an integral in $\mathbb{R}^2$. Calculate this last integral using polar coordinates.

Problem 2.3.2 Let $A = [0,1] \times [0,1]$.

a) Prove that the function $f(x,y) = \frac{|x-y|}{(x+y)^3}$ is not integrable in A.

b) Find out if the function $f(x,y) = \frac{1}{\sqrt{xy}}$ is integrable in A and, in that case, calculate the integral $\iint f(x,y)\, dxdy$.

c) Calculate $\iint_A x\,[1+x+y]\, dxdy$ where $[t]$ denotes the integer part of t, discussing before the integrability of the function.

Hint: a) Use the change of variables $x = y + t$ and use Fubini's theorem.

Problem 2.3.3 Using Tonelli-Fubini's theorem to justify all steps, evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos\frac{\pi y}{2x}\, dx\, dy\,.$$

Hint: Prove first that $g(x,y) = x^{-3/2}\cos\frac{\pi y}{2x} \geq 0$ on $A = \{(x,y) : 0 \leq y \leq x \leq 1\}$. Then apply Tonelli-Fubini's theorem.

Problem 2.3.4 Let us consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, with μ the counting measure.

a) Prove that $\mu \otimes \mu$ is the counting measure on $(\mathbb{N}\times\mathbb{N}, \mathcal{P}(\mathbb{N}\times\mathbb{N}))$.

b) Let us define the function

$$f(m,n) = \begin{cases} 1 & \text{if} \quad m = n\,, \\ -1 & \text{if} \quad m = n+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Check that $\int_{\mathbb{N}}(\int_{\mathbb{N}} f(m,n)\, d\mu(m))\, d\mu(n)$, and $\int_{\mathbb{N}}(\int_{\mathbb{N}} f(m,n)\, d\nu(n))\, d\mu(m)$ exist and are distinct and that $\int_{\mathbb{N}\times\mathbb{N}} |f(m,n)|\, d(\mu\otimes\mu)(m,n) = \infty$. What is the relevance of this result?

c) Do the same for the function

$$g(m,n) = \begin{cases} 1 + 2^{-m} & \text{if } m = n\,, \\ -1 - 2^{-m} & \text{if } m = n+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2.3.5 Let $(X, \mathcal{A})$ be a measurable space an let $f : X \longrightarrow [0,\infty]$ be a positive $\mathcal{A}$-measurable function. Let

$$A_f = \{(x,y) \in X \times \mathbb{R} :\ 0 \leq y \leq f(x)\}.$$

a) Prove that $A_f \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})$.

b) Given a σ-finite measure μ in $(X, \mathcal{A})$ prove that $\int_X f\, d\mu$ coincides with the product measure $\pi = \mu \otimes m$ of the set A_f, where m denotes Lebesgue measure in $\mathbb{R}$.

Hints: a) Prove it first for simple functions $s(x)$ in X and later for positive functions in X. b) Use the monotone convergence theorem.

Problem 2.3.6 Let $X = Y = [0,1]$, $\mathcal{A}_1\,,\ \mathcal{A}_2 = \mathcal{B}([0,1])$, μ the Lebesgue measure on $\mathcal{A}_1$, ν the counting measure on $\mathcal{A}_2$. In the measure space $(X \times Y, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu \otimes \nu)$ we consider the set $V = \{(x,y) : x = y\}$. Check that $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$. However

$$\int_Y d\nu \int_X \chi_V\, d\mu = 0\,, \qquad \int_X d\mu \int_Y \chi_V\, d\nu = 1\,.$$

What hypothesis of Fubini's theorem does not hold?

Hint: If $V_n = (I_1 \times I_1) \cup \cdots \cup (I_n \times I_n) \cup \{(1,1)\}$ being $I_j = [\frac{j-1}{n}, \frac{j}{n})$ $j = 1, 2, \ldots, n$, then $V = \cap_1^\infty V_n$.

Problem 2.3.7 Let $(X_k, \mathcal{A}_k, \mu_k)$ be σ-finite measure spaces, $k = 1, 2 \ldots, n$. Let $f_k : X_k \longrightarrow [0, \infty]$ be positive $\mathcal{A}_k$-measurable functions, $k = 1, 2 \ldots, n$.

a) Prove that the product function $h = f_1 f_2 \ldots f_n : X_1 \times \cdots \times X_n \longrightarrow [0, \infty]$ given by

$$h(x_1, \ldots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

is $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$-measurable and that

$$\int_{X_1 \times \cdots \times X_n} (f_1 f_2 \ldots f_n)\, d\mu_1 \otimes \cdots \otimes d\mu_n = \prod_{i=1}^n \int_{X_i} f_i\, d\mu_i\,. \tag{1}$$

b) Use this formula to compute the integral $\displaystyle\int_{\mathbb{R}^n} e^{-\|x\|^2}\, dx\,.$

c) Calculate again this integral using the formula for radial functions in Problem 2.2.26 and from this obtain the value of $\Omega_n = m(B_n)$, the n-dimensional Lebesgue measure of the unit ball B_n of $\mathbb{R}^n$.

d) Prove that part a) also holds when the functions $f_1, \ldots, f_k$ are not positive but $f_k \in L^1(\mu_k)$, $k = 1, 2 \ldots, n$.

Hints: a) Consider the functions $F_i(x_1, x_2, \ldots, x_n) := f_i(x_i)$ and use Fubini's theorem for positive functions. b) Use a) and problem 2.3.1. c) Use Euler's Gamma function and that $x\Gamma(x) = \Gamma(x+1)$. d) Use Fubini's theorem.

Problem 2.3.8 Let us consider the Lebesgue measure on $\mathbb{R}^2$. Let $A = [a,b] \times [c,d]$ and let f be continuous on A. Prove that

$$\int_A f\, dm = \int_a^b dx \int_c^d f(x,y)\, dy = \int_c^d dy \int_a^b f(x,y)\, dx\,.$$

Problem 2.3.9 Let

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{(x^2+y^2)^2}\,, & (x,y)\neq(0,0)\,,\\ 0\,, & (x,y)=(0,0)\,.\end{cases}$$

Check that

$$\int_0^1 dx\int_0^1 f(x,y)\,dy=\frac{\pi}{4}\,,\qquad \int_0^1 dy\int_0^1 f(x,y)\,dx=-\frac{\pi}{4}\,.$$

What hypothesis of Fubini's theorem does not hold?

Hint: $\frac{\partial}{\partial y}big(\frac{y}{x^2+y^2}\big)=\frac{x^2-y^2}{(x^2+y^2)^2}\big)$.

Problem 2.3.10 Let us define the function $f:[-1,1]\times[-1,1]\longrightarrow\mathbb{R}$ given by

$$f(x,y)=\begin{cases}\frac{xy}{(x^2+y^2)^2}\,, & (x,y)\neq(0,0)\,,\\ 0\,, & (x,y)=(0,0)\,.\end{cases}$$

Check that

$$\int_{-1}^1 dx\int_{-1}^1 f(x,y)\,dy=\int_{-1}^1 dy\int_{-1}^1 f(x,y)\,dx\,,$$

but however f is not integrable in $[-1,1]\times[-1,1]$. Why is relevant this exercise?

Problem 2.3.11 Sometimes, Fubini's Theorem can be used as a tool to show that a one variable integral converges to a certain value, by *transforming* the simple integral into a double one and, in a justified way, exchange order of integration. With this idea in mind and using that

$$\frac{1}{x}=\int_0^\infty e^{-tx}dt,$$

show that

$$\lim_{R\to\infty}\int_0^R\frac{\sin x}{x}\,dx=\frac{\pi}{2}\,.$$

Hint: Consider the function $f(x,t)=e^{-xt}\sin x$ defined in the set $(0,R)\times(0,\infty)$ and prove that

$$\int_0^R dx\int_0^\infty f(x,t)\,dt=\int_0^R\frac{\sin x}{x}\,dx<\infty\quad\text{but}\quad\int_0^\infty dt\int_0^R f(x,t)\,dx=\frac{\pi}{2}-\int_0^\infty\frac{e^{-Rt}(\cos R+t\sin R)}{1+t^2}\,dt.$$

Finally, using dominated convergence, prove that this last integral converges to zero as $R\to\infty$.

Problem 2.3.12

a) Prove that the function $f(x,y)=e^{-y}\sin 2xy$ is integrable in $A=[0,1]\times[0,\infty)$.

b) Prove that

$$\int_0^1 e^{-y}\sin 2xy\,dx=\frac{e^{-y}}{y}\,\sin^2 y\,,\qquad \int_0^\infty e^{-y}\sin 2xy\,dy=\frac{2x}{1+4x^2}\,.$$

c) Using Fubini's theorem, prove that:

$$\int_0^\infty e^{-y}\,\frac{\sin^2 y}{y}\,dy=\frac{1}{4}\,\log 5\,.$$

Problem 2.3.13 Let μ be the Lebesgue measure on $[0,1]$ and ν be the counting measure on $\mathbb{N}$. Let us define $G : [0,1] \times \mathbb{N} \longrightarrow \mathbb{R}$ by $G(x,n) = \left(\frac{x}{2}\right)^n$.

a) Prove that for $0 < a \leq 1$ we have that $G^{-1}((-\infty, a)) = \cup_n([0, 2a^{1/n}) \times \{n\})$.

b) Deduce that G is $\mu \otimes \nu$-measurable.

c) Use Fubini's theorem to prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)2^n} = 2\log 2 - 1.$$

Hint: b) Use Problem 1.1.13

Problem 2.3.14 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1\,, & \text{if } x \in [0,1] \cap \mathbb{Q},\ y \in [0,1]\,, \\ 0\,, & \text{if } x \in [0,1] \setminus \mathbb{Q},\ y \in [0,1]\,. \end{cases}$$

a) Prove that f is measurable with respect to Lebesgue σ-algebra.

b) Prove that $\displaystyle\iint_{[0,1]^2} f(x,y)\, dx\, dy = 0$.

Problem 2.3.15 Let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} 1\,, & \text{if } xy \in \mathbb{Q}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

a) Prove that f is measurable with respect to Lebesgue σ-algebra.

b) Prove that $\displaystyle\iint_{[0,1]^2} f(x,y)\, dxdy = 0$.

Problem 2.3.16 Let us consider the measure space $([0,1] \times [0,1], \mathcal{M}, m_2)$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and m_2 is the two-dimensional Lebesgue measure. Given $E \in \mathcal{M}$, let us denote

$$E_x = \{y \in [0,1] :\ (x,y) \in E\}\,, \qquad E^y = \{x \in [0,1] :\ (x,y) \in E\}\,.$$

Let m_1 denote Lebesgue measure on $[0,1]$. Prove that if $E \in \mathcal{M}$ verifies that $m_1(E_x) \leq 1/2$ for almost all $x \in [0,1]$, then

$$m_1(\{y \in [0,1] :\ m_1(E^y) = 1\}) \leq \frac{1}{2}\,.$$

Hint: Apply Fubini's theorem to the function $f = \chi_E$ and consider the set $A = \{y \in [0,1] : m_1(E^y) = 1\}$.

Problem 2.3.17 Let $f \in L^1(0,\infty)$. Given $\alpha > 0$, let us define $g_\alpha(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$ for $x > 0$. Check that $\alpha \int_0^y g_\alpha(x) dx = g_{\alpha+1}(y)$ for $y > 0$.

Hint: Check that you can apply Tonelli-Fubini's theorem.

Problem 2.3.18 Let f and g be Lebesgue integrable functions on $[0,1]$, and let F and G be the integrals

$$F(x)=\int_0^x f(t)\,dt\,,\qquad G(x)=\int_0^x g(t)\,dt\,.$$

Use Fubini's theorem to prove that

$$\int_0^1 F(x)g(x)\,dx = F(1)G(1)-\int_0^1 f(x)G(x)\,dx\,.$$

Problem 2.3.19* Apply Fubini's theorem to obtain the following recurrence formula for n-dimensional measure Ω_n of the unit ball B_n of $\mathbb{R}^n$:

$$\Omega_n=\sqrt{\pi}\,\Omega_{n-1}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\,.$$

Hint: $\Omega_n=\int_{-1}^1 m_{n-1}(B_{x_1})\,dx_1$ where $B_{x_1}=\{\bar{x}\in\mathbb{R}^{n-1}:\|\bar{x}\|<(1-x_1^2)^{1/2}\}$. Relate $m_{n-1}(B_{x_1})$ with Ω_{n-1} and use the Euler's β-function $\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt$ and the formula $\beta(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, where $\Gamma(x)=\int_0^\infty t^{x-1}e^{-x}dx$ is the Euler Γ-function.

Problem 2.3.20* Given $x\in\mathbb{R}^n\setminus\{0\}$, let us consider its polar coordinates (r,x') where $r=\|x\|\in(0,\infty)$, $x'=x/\|x\|\in S_{n-1}=\{x\in\mathbb{R}^n:\ \|x\|=1\}$. The mapping

$$\varphi:\mathbb{R}^n\setminus\{0\}\longrightarrow(0,\infty)\times S_{n-1}\qquad\text{given by }\varphi(x)=(r,x')$$

is a bijection. Prove that

a) If μ is the image measure under φ of the Lebesgue measure on $\mathbb{R}^n\setminus\{0\}$, then

$$\mu(E\times U)=\sigma(U)\int_E r^{n-1}dr\,,\qquad\text{for all borel sets } E\subseteq(0,\infty),\ U\subseteq S_{n-1}\,.$$

b) If $f:\mathbb{R}^n\setminus\{0\}\longrightarrow[0,\infty]$ is a positive measurable function, then

$$\int_{\mathbb{R}^n} f(x)\,dx=\int_0^\infty r^{n-1}dr\int_{S_{n-1}} f(rx')\,d\sigma(x')$$

where σ is the $(n-1)$-dimensional Lebesgue measure on S_{n-1}.

c) Given $f(x)=|x_1x_2\cdots x_n|$, use Fubini's theorem to obtain a recurrence formula relating $I_n=\int_{B_n} f(x)\,dx$ with I_{n-1}. Deduce the value of I_n.

d) Apply parts b) and c) to evaluate $J_n=\int_{S_{n-1}} f(x')\,d\sigma(x')$, .

Hints: a) For each fixed Borel set $U\subset S_{n-1}$, as a consequence of Caratheodory-Hopf's theorem, it suffices to prove that both sides of the identity coincide for semi-intervals $E=[a,b)$. b) Observe that $f=f\circ\varphi\circ\varphi^{-1}$ and use first problem 2.2.21, part a) and later Fubini's theorem.