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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.4: Decomposition of measures

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2 Integration Theory

2.4. Decomposition of measures

Problem 2.4.1 Let μ, λ be measures defined on the same σ-algebra. Prove that if we have at the same time $\lambda \perp \mu$ and $\lambda \ll \mu$, then $\lambda \equiv 0$.

Problem 2.4.2 Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}([0,1])$, m the Lebesgue measure on $\mathcal{M}$, μ the counting measure on $\mathcal{M}$.

a) Prove that $m \ll \mu$ but $dm \neq f\, d\mu$ for all f.

b) Prove that m has not Radon-Nikodym decomposition with respect to μ.

c) What hypothesis fails to apply in Radon-Nikodym theorem?

Hints: a) Suppose that there exists such an f. Then, there exists x_0 such that $f(x_0) \neq 0$. Consider the set $A = \{x_0\}$. b) How must be a measure which is mutually orthogonal with μ?

Problem 2.4.3 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N}$ be a σ-subalgebra of $\mathcal{M}$ and let ν be the restriction of μ to $\mathcal{N}$. If $0 \leq f \in L^1(\mu)$ prove that there exists g $\mathcal{N}$-measurable, $0 \leq g \in L^1(\nu)$, such that $\int_E f\, d\mu = \int_E g\, d\nu$ for all $E \in \mathcal{N}$. Besides, g is unique module alterations in ν-null sets.

Hint: Consider the measure $\lambda(E) = \int_E f\, d\mu$ for $E \in \mathcal{N}$. The function g is called the *conditioned expected value* $E(f|\mathcal{N})$ *of* f *with respect to* $\mathcal{N}$.

Problem 2.4.4 Let μ and ν be finite positive measures on the measurable space $(X, \mathcal{A})$. Show that there is a nonnegative measurable function f on X such that for all $A \in \mathcal{A}$

$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu\,.$$

Hint: The statement is equivalent to $\mu(A) = \int_A f\, d(\mu + \nu)$.

Problem 2.4.5 Let m be the Lebesgue measure on the real line $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$ define

$$\mu(E) = \int_E \frac{1}{1 + x^2}\, dm(x)\,.$$

a) Show that $m \ll \mu$.

b) Compute the Radon-Nikodym derivative $h = dm/d\mu$.

Hints: a) Use Problem 2.1.1. b) Given a Lebesgue-measurable set E, consider the function $f(x) = \frac{1}{1+x^2}\, \chi_E(x)$ and use problem 2.2.21.

Problem 2.4.6 Let m be the Lebesgue measure on the real line $\mathbb{R}$. Consider a measurable function $f : \mathbb{R} \to [0, \infty]$ such that f and $1/f$ are Lebesgue integrable on each bounded subset of $\mathbb{R}$. For each Lebesgue measurable subset of $\mathbb{R}$ define

$$\mu(E) = \int_E f(x)\, dm(x)\,.$$

a) Show that $m \ll \mu$.

b) Compute the Radon-Nikodym derivative $h = dm/d\mu$.

Hint: Use the argument in the previous exercise.

Problem 2.4.7 Let us consider the increasing function $F(x) = \max\{0, x + [x]\}$, where $[x]$ denotes the integer part of x. Let μ_F be the Borel-Stieltjes measure associated to F.

a) Calculate $\mu_F((0,5])$, $\mu_F([4,8])$ and $\mu_F([3,7))$.

b) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure.

Hint: b) Consider, for example, the set $A = \{1\}$.

Problem 2.4.8 Let μ_F be the Borel-Stieltjes measure associated to the increasing function

$$F(x) = \begin{cases} 0\,, & \text{if } x < 0\,, \\ x\,, & \text{if } 0 \le x < 1\,, \\ 1\,, & \text{if } 1 \le x\,. \end{cases}$$

a) Prove that $\mu_F \ll m$, being m the Lebesgue measure.

b) Is there the Radon-Nikodym derivative of μ_F with respect to m?. If so, find it.

c) Let μ be the measure that counts rational numbers in $[0,1]$, that is to say that $\mu(A) = \operatorname{card}(A \cap [0,1] \cap \mathbb{Q})$. Prove that $\mu_F \perp \mu$.

Hints: a) Given $I = [a,b)$, prove that $\mu_F(I) \le m(I)$ since $x - F(x)$ is increasing. Apply Caratheodory-Hopf's theorem. b) Use problem 2.2.25. c) Consider the set $[0,1] \cap \mathbb{Q}$.

Problem 2.4.9 Let us define the increasing function

$$F(x) = \begin{cases} e^x\,, & \text{if } x < 0\,, \\ 2 + \arctan x\,, & \text{if } x \ge 0\,, \end{cases}$$

and let μ_F be the associated Borel-Stieltjes measure.

a) Calculate $\mu_F((0,1])$, $\mu_F((-2,0])$, $\mu_F(0)$ and $\mu_F(\mathbb{R})$.

b) Prove that $\mu_F = f(x)\,dx + \delta_0$, for certain $f \ge 0$, and being δ_0 the δ-Dirac measure at $x = 0$.

c) Is true that $\mu_F \ll dx$?

Hint: b) Observe that $\delta_0 = \mu_H$ with H the Heaviside function: $H = \chi_{[0,\infty)}$ and apply Exercise 2.2.25 to μ_{F-H}.

Problem 2.4.10 Let us consider the increasing function

$$F(x) = \begin{cases} 0\,, & \text{if } x < 0\,, \\ x^2/2\,, & \text{if } 0 \le x < 1\,, \\ 1\,, & \text{if } 1 \le x\,, \end{cases}$$

and let μ_F be the Borel-Stieltjes measure associated to F.

a) Prove that μ_F is not absolutely continuous with respect to Lebesgue measure m.

b) Find the Radon-Nikodym decomposition of μ_F with respect to m.

Hint: b) Use problem 2.2.25 on $(-\infty,1)$ and on $(1,\infty)$.