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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.5: L^p-spaces

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2 Integration Theory

2.5. L^p-spaces

Problem 2.5.1 Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be functions such that

$$\varphi_i \in L^{p_i}(X, \mathcal{A}, \mu), \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \leq 1 .$$

Then $\varphi_1\varphi_2\cdots\varphi_k \in L^p(X, \mathcal{A}, \mu)$ and $\|\varphi_1\varphi_2\cdots\varphi_k\|_p \leq \|\varphi_1\|_{p_1}\|\varphi_2\|_{p_2}\cdots\|\varphi_k\|_{p_k}$.

Hint: If $a_1, \cdots, a_k \geq 0$ and $\lambda_1 + \cdots \lambda_k = 1$, then $a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_k^{\lambda_k} \leq \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k$.

Problem 2.5.2 Let $0 < p < r < q \leq \infty$ and let $\varphi \in L^p(X, \mathcal{A}, \mu) \cap L^q(X, \mathcal{A}, \mu)$.

a) Prove that $\varphi \in L^r(X, \mathcal{A}, \mu)$ and

$$\|\varphi\|_r \leq \|\varphi\|_p^\theta \|\varphi\|_q^{1-\theta}, \qquad \text{where } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q} .$$

b) Prove also that $L^r(\mu) \subset L^p(\mu) + L^q(\mu)$.

c) Prove that $\lim_{r\to\infty} \|\varphi\|_r = \|\varphi\|_\infty$.

Hints: a) If $q = \infty$, then $|\varphi|^r = |\varphi|^{r-p}|\varphi|^p \leq \|\varphi\|_\infty^{r-p}|\varphi|^p$ and $\frac{1}{r} = \frac{\theta}{p}$. If $q < \infty$, then $\frac{p}{\theta r}$ and $\frac{q}{(1-\theta)r}$ are conjugate exponents and $|\varphi|^r = |\varphi|^{\theta r}|\varphi|^{(1-\theta)r}$. Apply Hölder's inequality. b) If $A = \{x \in X : \ |\varphi(x)| \leq 1\}$, then $\varphi = \varphi\chi_A + \varphi\chi_{A^c}$. c) By letting $r \to \infty$ in $\|\varphi\|_r \leq \|\varphi\|_p^\theta \|\varphi\|_\infty^{1-\theta}$ deduce that $\limsup_{r\to\infty} \|\varphi\|_r \leq \|\varphi\|_\infty$. Also, we can suppose that $\|\varphi\|_\infty > a > 0$. Use Markov's inequality to deduce that $\|\varphi\|_r \geq a\,\mu(\{x : |\varphi(x)| > a\})^{1/r}$ and by letting $r \to \infty$ and $a \to \|\varphi\|_\infty$ deduce that $\liminf_{r\to\infty} \|\varphi\|_r \geq \|\varphi\|_\infty$.

Problem 2.5.3 Let $(X, \mathcal{A}, \mu)$ be a measure space. For some measures the relation $p < q$ implies $L^p \subset L^q$. For others the relationship is reversed and there are some measures for which L^p does no contain L^q for $p \neq q$. Give examples of these situations:

a) If $\mu(X) < \infty$ and $1 \leq p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq \|f\|_q\,\mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

b) If $0 < p < q \leq \infty$, then $\ell^p \subset \ell^q$ and $\|x_n\|_q \leq \|x_n\|_p$.

c) Show that $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), m) \not\subseteq L^q(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ for $p \neq q$.

Hints: a) Use Hölder's inequality. b) Use part *a)* of problem 2.5.2. c) Consider the function $f(x) = |x(\log^2|x| + 1)|^{-1/p}$.

Problem 2.5.4 Let $(X, \mathcal{A}, \mu)$ be a measure space.

i) Prove that Hölder's inequality holds for the exponents $p = 1$ and $q = \infty$: If f and g are measurable functions on X, then $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.

ii) If $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, prove that $\|fg\|_1 = \|f\|_1\|g\|_\infty$ iff $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

iii) Prove that if $f \in L^p(\mu)$ and $g \in L^\infty(\mu)$, then $fg \in L^p(\mu)$ and $\|fg\|_p \leq \|f\|_p\|g\|_\infty$. When equality holds in this inequality?

iv) Prove that $\|\cdot\|_\infty$ is a norm on $L^\infty(\mu)$.

v) Prove that if $\mu(X) < \infty$ and $f \in L^\infty(\mu)$, then $f \in \cap_{p\geq 1} L^p(\mu)$. Prove that the reverse statement is false.

vi) Let $f \in L^\infty(\mu)$ and $\{f_n\}$ be a sequence in $L^\infty(\mu)$. Prove that $\|f_n - f\|_\infty \to 0$ if and only if there exists $E \in \mathcal{A}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

vii) The simple functions are dense in L^∞ if $\mu(X) < \infty$: Each $f \in L^\infty$ can be approximated by a sequence of simple functions $\{s_n\} \subset L^\infty(\mu)$.

Hint: v) Consider the function $f(x) = \log x$ on $X = (0, 1]$.

Problem 2.5.5 Let $1 \leq p < \infty$.

a) Show that if $\varphi \in L^p(\mathbb{R}^N)$ and φ is uniformly continuous, then $\lim_{|x|\to\infty} \varphi(x) = 0$.

b) Show that this is false if one only assumes that φ is continuous.

Hint: a) Prove it by contradiction: if $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ is such that $|x_n| \to \infty$ and $|\varphi(x_n)| \geq \delta > 0$ for every n, then the uniform continuity of φ implies the existence of $R > 0$ such that $|\varphi(x)| \geq \delta/2$ in $B(x_n, R)$. Show that this yields $\int_{\mathbb{R}^N} |\varphi|^p dx = \infty$. b) Consider the function $\varphi(x) = \sum_{n=1}^\infty f_n(x - n)$, where

$$f_n(x) = \begin{cases} nx + 1\,, & \text{if } -1/n \leq x \leq 0\,, \\ 1 - nx\,, & \text{if } 0 \leq x \leq 1/n\,, \\ 0\,, & \text{if } x \notin (-1/n, 1/n)\,. \end{cases}$$

Problem 2.5.6 Suppose that $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots$ and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. What relation exists between f and g?

Problem 2.5.7 Suppose $\mu(X) = 1$, and suppose f and g are positive measurable functions on X such that $fg \geq 1$. Prove that

$$\int_X f\, d\mu \cdot \int_X g\, d\mu \geq 1\,.$$

Hint: Use Cauchy-Schwarz ineguality.

Problem 2.5.8 Suppose $\mu(X) = 1$ and $h : X \longrightarrow [0, \infty]$ is measurable. If $A := \int_X h\, d\mu$, prove that

$$\sqrt{1 + A^2} \leq \int_X \sqrt{1 + h^2}\, d\mu \leq 1 + A\,.$$

If μ is Lebesgue measure on $[0, 1]$ and h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general X) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Hint: The first inequality follows from Jensen's inequality. The second one follows from the inequality $\sqrt{1 + x^2} \leq 1 + x$ for $x \geq 0$.

Problem 2.5.9 Let f be a complex function, $f \neq 0$. Let us define the function $\varphi(p) = \|f\|_p^p$ for $0 < p < \infty$ and let $E = \{p : \varphi(p) < \infty\} = \{p : f \in L^p(\mu)\}$. Prove that

a) If $r < p < s$ and $r, s \in E$, then $p \in E$.

b) $\log \varphi$ is convex in E.

c) Part a) implies that E is connected. Is E necessarily open? and closed? Can E be constituted by a single point? Can E be a any connected subset of $(0, \infty)$?

d) If $r < p < s$, then $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$.

Hints: a) $t^p \leq \max(t^r, t^s) \leq t^r + t^s$. b) If $p = \lambda r + (1-\lambda)s$ with $0 < \lambda < 1$, apply Hölder's inequality (with the conjugate exponents $\alpha = 1/\lambda$ and $\beta = 1/(1-\lambda)$) to bound $\varphi(p)$ in terms of $\varphi(r)$ and $\varphi(s)$. d) Apply part b).

Problem 2.5.10* Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e. $\mu(X) = 1$.

a) Prove that if φ is strictly convex: $\varphi(\lambda x + (1-\lambda)y) < \lambda\varphi(x) + (1-\lambda)\varphi(y)$ for $0 < \lambda < 1$, then equality holds in Jensen's inequality,

$$\varphi\Big(\int_X f \, d\mu\Big) \leq \int_X (\varphi \circ f) \, d\mu \,, \qquad \text{for } f \in L^1(\mu)\,,$$

if and only if f is constant almost everywhere.

b) If $0 < p < q \leq \infty$ prove that $\|f\|_p \leq \|f\|_q$.

c) Use part a) to prove that $\|f\|_p = \|f\|_q$ if and only if f is constant almost everywhere.

d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\Big(\int_X \log|f| \, d\mu\Big)$$

if $\exp(-\infty)$ is defined to be 0.

Hints: a) If $f \neq 0$ a.e., then there exists $c \in \mathbb{R}$ such that $A = \{x : |f(x)| > c\}$ has $0 < \mu(A) < 1$. Take $\lambda = \mu(A)$, $x = \frac{1}{\lambda}\int_A f \, d\mu$, $y = \frac{1}{1-\lambda}\int_{A^c} f \, d\mu$ and apply Jensen's inequality. To bound $\varphi(x)$ and $\varphi(y)$ apply again Jensen's inequality. Finally, deduce that Jensen's inequality for this f is strict. b) Apply Jensen's inequality to the convex function $\varphi(x) = x^t$ with $t = q/p > 1$. c) $\varphi(x) = x^t$ is strictly convex. d) Apply Jensen's inequality with $\varphi(x) = -\log x$ and use that $\log x \leq x - 1$ for $x \in (0, \infty)$ and that $(t^p - 1)/t \to \log t$ as $p \to 0$. Use a convergence theorem.

Problem 2.5.11** Suppose $1 < p < \infty$, $f \in L^p((0,\infty), \mathcal{B}, m)$ and let us define

$$F(x) = \frac{1}{x}\int_0^x f(t) \, dt \qquad (0 < x < \infty).$$

a) Prove that the mapping $f \to F$ carries L^p into L^p and more concretely, prove Hardy's inequality:

$$\|F\|_p \leq \frac{p}{p-1} \, \|f\|_p \,.$$

b) Prove that equality holds in Hardy's inequality iff $f = 0$ almost everywhere.

c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Hints: a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx\,.$$

Note that $xF' = f - F$ and apply Hölder's inequality to $\int F^{p-1}f$. Then derive the general case. b) If equality holds for $f \geq 0$ deduce that we must have equality in

$$\int_0^\infty F^p(x)\,dx = q\int_0^\infty f(x)F^{p-1}\,dx \leq q\|f\|_p\Big(\int_0^\infty F^p(x)\,dx\Big)^{1/q}$$

and therefore that $\exists\ \alpha \geq 0$ such that $\alpha f^p = F^p$, and from this that f is constant a.e. c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. d) If $f \in L^1$ and $f \neq 0$ a.e., then $\exists\ x_0$ such that $\int_0^{x_0} f(t)\,dt > 0$.

Problem 2.5.12 Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < \infty$ and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $L^p(\mu)$ such that $f_n \to f$ almost everywhere, as $n \to \infty$.

a) If, for some $M \geq 0$, $\|f_n\|_p \leq M$ for all $n \in \mathbb{N}$, then $f \in L^p(\mu)$ and

$$\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p\,.$$

b) If, for some $F \in L^p(\mu)$, $|f_n(x)| \leq |F(x)|$ for all $n \in \mathbb{N}$ and almost every $x \in X$, then $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

c) Prove that b) is false for $p = \infty$.

Hints: a) Use Fatou's lemma. b) Use dominated convergence theorem. c) Consider the sequence $f_n = \chi_{(0,1/n)}$ in $(0, 1)$.

Problem 2.5.13* Let $0 < p < \infty$ and $f, f_n \in L^p(X, \mathcal{A}, \mu)$.

a) If $1 \leq p \leq \infty$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$, prove that $\|f_n\|_p \to \|f\|_p$.

b) Let $c_p = \max\{1, 2^{p-1}\}$. Prove that

$$|a - b|^p \leq c_p\,(|a|^p + |b|^p)$$

for arbitrary complex numbers a and b.

c) If $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$ prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

d) Prove that the conclusion of c) is false if the hypothesis $\|f_n\|_p \to \|f\|_p$ is removed, even if $\mu(X) < \infty$.

e) Prove that the conclusion of c) is false if $p = \infty$

Hint: a) Prove that $\big|\|f\|_p - \|g\|_p\big| \leq \|f - g\|_p$ for $f, g \in L^p(\mu)$. b) Prove the cases $0 < p \leq 1$ and $1 < p < \infty$ separately. For the first one, consider the function $\phi(x) = (x+y)^p - x^p - y^p$ for $x \geq 0$ and fixed $y \geq 0$ and prove that φ is decreasing. For the second case, consider the function $\psi(x) = 2^{p-1}(x^p + y^p) - (x+y)^p$ for $x \geq 0$ and fixed $y \geq 0$ and prove that ψ has an absolute minimum when $x = y$. c) Consider the function $h_n = c_p\,(|f|^p + |f_n|^p) - |f - f_n|^p$ and use Fatou's lemma as in the proof of the dominated convergence theorem.