uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.3: Laplace transform

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2 Integrals depending on a parameter

3.3. Laplace transform

Problem 3.3.1 Given $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, calculate the Laplace's transform of the following functions:

1) $f(t) = 1\,,$	2) $f(t) = t^n\,,$
3) $f(t) = \cos at\,,$	4) $f(t) = \sin at\,,$
5) $f(t) = \frac{\sin at}{t}\,,$	6) $f(t) = e^{at}\,,$
7) $f(t) = t^n e^{at}\,,$	8) $f(t) = e^{at}\sin bt\,,$
9) $f(t) = e^{at}\cos bt\,,$	10) $f(t) = e^{at}\sinh bt\,,$
11) $f(t) = e^{at}\cosh bt\,,$	12) $f(t) = t\cos at\,,$
13) $f(t) = t^2 e^{-t}\cos t\,,$	14) $f(t) = t\int_0^t e^{-x}\sin x\,dx\,,$
15) $f(t) = \cos^3 t\,,$	16) $f(t) = \begin{cases} 0\,, & \text{if } 0 \le t \le 1\,, \\ (t-1)^k, & \text{if } t > 1\,, \end{cases} \quad (k \in \mathbb{N})\,.$

Problem 3.3.2 a) A function f is said periodic with period $P > 0$, if $f(t) = f(P+t)$ for all $t \in (0, \infty)$. Prove that the Laplace transform transform of a periodic function with period $P > 0$ verifies

$$\mathcal{L}[f](z) = \frac{1}{1 - e^{-Pz}} \int_0^P e^{-zt} f(t)\,dt\,.$$

b) Calculate the Laplace transform of the function $f(t) = t - [t]$, being $[t]$ the integer part of t.

Problem 3.3.3 Calculate the inverse Laplace transform of the following functions:

$$1)\ F(z) = \frac{e^{3z}}{z^3 + 4z}\,, \qquad 2)\ F(z) = \frac{e^{-3z}}{z^3 + 4z}\,,$$

$$3)\ F(z) = \frac{1}{(z^2 + a^2)^2}\,, \qquad 4)\ F(z) = \frac{z}{(z^2 + a^2)^2}\,,$$

$$5)\ F(z) = \log(1 + 1/z)\,.$$

Problem 3.3.4 a) If $F(z)$ denotes the Laplace transform of f and $f \in L^1(0, \infty)$, prove the identity

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(s)\,ds\,.$$

b) Use this identity to calculate

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx\,.$$

Problem 3.3.5 Calculate the integral $f(t) = \displaystyle\int_0^\infty \frac{\cos xt}{1 + x^2}\,dx\,.$

Hint: Calculate $F(z) = \mathcal{L}[f](z)$ using Fubini's theorem. Then, calculate the inverse Laplace transform of $F(z)$.

Problem 3.3.6 Solve the following initial value problems, obtaining first the Laplace transform $Y(z)$ of the solution $y(t)$ and then, anti-transforming $Y(z)$:

$$1)\ \begin{cases} y' - 5y = \cos 3t\,, \\ y(0) = 1/2\,, \end{cases} \qquad 2)\ \begin{cases} y'' + 16y = \cos 4t\,, \\ y(0) = 0\,,\ y'(0) = 1\,, \end{cases}$$

$$3)\ \begin{cases} y'' - 6y' + 9y = t^2 e^{3t}\,, \\ y(0) = 2\,,\ y'(0) = 6\,, \end{cases} \qquad 4)\ \begin{cases} y'' + 4y' + 6y = 1 + e^{-t}\,, \\ y(0) = 0\,,\ y'(0) = 0\,. \end{cases}$$

Problem 3.3.7 Solve the initial value problem for the system of differential equations:

$$\begin{cases} x' - 6x + 3y = 8e^t\,, \\ y' - 2x - y = 4e^t\,, \\ x(0) = -1\,,\ y(0) = 0\,. \end{cases}$$

Problem 3.3.8 Solve the following initial value problems:

$$1)\ \begin{cases} y'' + y' = \begin{cases} t+1 & \text{if } 0 < t < 1\,, \\ 3-t & \text{if } t > 1\,, \end{cases} \\ y(0) = -1,\ y'(0) = 0\,. \end{cases} \qquad 2)\ \begin{cases} y'' + 4y = \begin{cases} \cos 2t & \text{if } 0 < t < 2\pi\,, \\ 0 & \text{if } t > 2\pi\,, \end{cases} \\ y(0) = y'(0) = 0\,. \end{cases}$$

Problem 3.3.9 Let us consider the differential equation $tx'' + 2x' + tx = 0$ for $t > 0$.

a) Find the differential equation that verifies the Laplace transform $X(z)$ of $x(t)$ with the initial data $x(0) = 1$.

b) Solve the differential equation for $X(z)$ using the property $\lim_{z \to +\infty} X(z) = 0$.

c) Calculate the Laplace anti-transform $x(t)$.

Problem 3.3.10 Solve the following initial value problems:

$$1)\ \begin{cases} y'' = \delta_a\,, \\ y(0) = 0\,,\ y'(0) = 0\,. \end{cases} \qquad 2)\ \begin{cases} y' + 8y = \delta_1 + \delta_2\,, \\ y(0) = 0\,. \end{cases}$$

Problem 3.3.11 a) Let us consider the Volterra integral equation

$$y(t) + \int_0^t k(t-x)\, y(x)\, dx = f(t)\,,$$

where f and k are known functions and $y(t)$ is the unknown function. Calculate the Laplace transform of $y(t)$ in terms of the transforms of k and f.
b) As an application, solve the Volterra equation

$$y(t) - 2\int_0^t \cos(t-x)\, y(x)\, dx = e^{2t}\,.$$

Problem 3.3.12 Solve, for $\omega \neq \omega_0$, the initial value problem

$$\begin{cases} x'' + \omega_0^2\, x = k \sin \omega t\,, \quad \text{if } t > 0\,, \\ x(0) = x'(0) = 0\,, \end{cases}$$

which describes the forced oscillations of a mass in a not damped spring. What happens if $\omega = \omega_0$? Explain physically the obtained results.

LAPLACE TRANSFORMS TABLE

$$f(t) = 1\,, \qquad \mathcal{L}[f](z) = \frac{1}{z} \qquad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^n \quad (n \in \mathbb{N})\,, \qquad \mathcal{L}[f](z) = \frac{n!}{z^{n+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^a \quad (a > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(a+1)}{z^{a+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2 + a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2 + a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \frac{\sin at}{t} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \arctan \frac{a}{z} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = e^{at} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{1}{z - a} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at} t^b \quad (a \in \mathbb{R},\, b > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(b+1)}{(z-a)^{b+1}} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at} \sin bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{b}{(z-a)^2 + b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at} \cos bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z - a}{(z-a)^2 + b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = \sinh at = \frac{e^{at} - e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2 - a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \cosh at = \frac{e^{at} + e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2 - a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \delta_a \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = e^{-az}\,,$$

$$f(t) = \sin at - at \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2a^3}{(z^2 + a^2)^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2az}{(z^2 + a^2)^2} \quad (\operatorname{Re} z > 0)\,.$$