Problem 1 (2,5 points)

a) **(1 point)** Let $X = \{a,b,c,d\}$. Construct the σ-algebra generated by $\mathcal{E}_1 = \{\{a\}\}$ and $\mathcal{E}_2 = \{\{a\},\{b\}\}$.

b) **(1,5 points)** Let $E \in \mathcal{A}$ be a fixed measurable subset of X. We define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{A}$. Using that μ is a measure in X, prove that μ_E is also a measure in X.

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a) To construct them we must add the necessary subsets so that the σ-algebra properties are verified: $\mathcal{A}_{\mathcal{E}_1} = \{\varnothing, \{a\}, \{b,c,d\}, X\}$, $\mathcal{A}_{\mathcal{E}_2} = \{\varnothing, \{a\}, \{b\}, \{b,c,d\}, \{a,c,d\}, \{a,b\}, \{c,d\}, X\}$.

b) Since μ is a measure: $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$;

Let $\{A_j\}_{j=1}^\infty$ be a disjoint countable collection for sets in $\mathcal{A}$. Then the collection $\{A_j \cap E\}_{j=1}^\infty$ is also disjoint and, as μ is a measure,

$$\mu_E\Big(\bigcup_{j=1}^\infty A_j\Big) = \mu\Big(\big(\bigcup_{j=1}^\infty A_j\big) \cap E\Big) = \mu\Big(\bigcup_{j=1}^\infty (A_j \cap E)\Big) = \sum_{j=1}^\infty \mu(A_j \cap E) = \sum_{j=1}^\infty \mu_E(A_j)\,.$$

Problem 2 (2,5 points)

a) (1 point) State the monotone convergence theorem.

b) (1.5 points) Prove that the function $f(x) = \frac{1}{\sqrt{x}}$ if $x \in (0,1]$, and $f(0) = 0$, is Lebesgue-integrable in $[0,1]$ and calculate its integral.

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a) Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions such that

$$0 \le f_1(x) \le f_2(x) \le \cdots \le \infty\,, \qquad \forall\, x \in X\,.$$

Then

$$\int_X \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_X f_n\,.$$

b) As $\frac{1}{\sqrt{x}}\,\chi_{[1/N,1]}(x) \nearrow \frac{1}{\sqrt{x}}\,\chi_{(0,1]}(x) = f(x)$ when $N \to \infty$ for $x \in [0,1]$, by the monotone convergence theorem,

$$\int_0^1 f(x)\,dx = \lim_{N\to\infty} \int_0^1 f(x)\,\chi_{[1/N,1]}(x)\,dx = \lim_{N\to\infty} \int_{1/N}^1 f(x)\,dx\,.$$

But $f(x)$ is continuous in the bounded interval $[1/N,1]$ and so it is Riemann-integrable in $[1/N,1]$ and its Lebesgue integral coincide with its Riemann integral, and to compute it we can use Barrow's rule. Hence,

$$\int_1^\infty f(x)\,dx = \lim_{N\to\infty} \int_{1/N}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty} [2\sqrt{x}\,]_{x=1/N}^{x=1} = \lim_{N\to\infty} 2\Big(\frac{1}{\sqrt{N}} - 1\Big) = 2.$$

Problem 3 (3 points)

- **a) (1 point)** State the dominated convergence theorem.
- **b) (2 points)** Using this last theorem, compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx\,.$$

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a) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of complex measurable functions such that $f_n(x) \to f(x)$ as $n \to \infty$ a.e. on X. If there exists a function $F \in L^1(\mu)$ such that

$$|f_n(x)| \le F(x)\,, \qquad \forall\, n \in \mathbb{N},\ a.e.\, x \in X,$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu\,.$$

b) Let $f_n(x) = (1 - \frac{x}{2})^n e^{x/2} \chi_{[0,n]}(x)$. As $\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$, we have $\lim_{n\to\infty} f_n(x) = e^{-x/2}$. Hence, we guess that:

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx = \lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty \left(\lim_{n\to\infty} f_n(x)\right) dx = \int_0^\infty e^{-x/2} dx = 2.$$

To prove it, we will show that $|f_n(x)| \le e^{-x/2} \in L^1(0,\infty)$ and then our conjecture will be a consequence of the dominated convergence theorem. To do that it is enough to prove that $(1 - \frac{x}{n})^n \le e^{-x}$ if $x \in [0, n]$. This inequality is equivalent to $n \log(1 - \frac{x}{n}) \le -x$. If we define $F(x) := x + n \log(1 - \frac{x}{n})$ for $x \in [0, n]$, then we must prove that $F(x) \le 0$ for $x \in [0, n]$. But

$$F'(x) = 1 - \frac{1}{1 - \frac{x}{n}} = -\frac{x/n}{1 - \frac{x}{n}} \le 0 \implies F \text{ is decreasing} \implies F(x) \le F(0) = 0\,.$$

Problem 4 (2 points) Let us consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, with μ the counting measure and the product measure space $(\mathbb{N} \times \mathbb{N}, \mathcal{P}(\mathbb{N} \times \mathbb{N}), \mu \otimes \mu)$. Let us define the function

$$g(m,n) = \begin{cases} 1 + 2^{-m} & \text{if} \qquad m = n\,, \\ -1 - 2^{-m} & \text{if} \qquad m = n+1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Check that $\int_{\mathbb{N}} (\int_{\mathbb{N}} f(m,n)\, d\mu(m))\, d\mu(n)$, and $\int_{\mathbb{N}} (\int_{\mathbb{N}} f(m,n)\, d\nu(n))\, d\mu(m))$ exist and are distinct and that $\int_{\mathbb{N}\times\mathbb{N}} |f(m,n)|\, d(\mu \otimes \mu)(m,n) = \infty$. What is the relevance of this result?

For fixed n we have:

$$\int_{\mathbb{N}} f(m,n)\, d\mu(m) = \sum_{m=1}^{\infty} f(m,n) = f(n,n) + f(n+1,n) = 1 + 2^{-n} - 1 - 2^{-n-1} = 2^{-n-1}\,,$$

and, for fixed m,

$$\begin{aligned}\int_{\mathbb{N}} f(m,n)\, d\mu(n) = \sum_{n=1}^{\infty} f(m,n) &= \begin{cases} f(1,1)\,, & \text{if } m=1\,,\\ f(m,m-1)+f(m,m)\,, & \text{if } m\geq 2\,,\end{cases}\\ &= \begin{cases} 1+2^{-1}\,, & \text{if } m=1\,,\\ -1-2^{-m}+1+2^{-m}\,, & \text{if } m\geq 2\,.\end{cases} = \begin{cases} 3/2\,, & \text{if } m=1\,,\\ 0\,, & \text{if } m\geq 2\,.\end{cases}\end{aligned}$$

Hence,

$$\int_{\mathbb{N}} \Big(\int_{\mathbb{N}} f(m,n)\, d\mu(m) \Big)\, d\mu(n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

and

$$\int_{\mathbb{N}} \Big(\int_{\mathbb{N}} f(m,n)\, d\mu(n) \Big)\, d\mu(m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \frac{3}{2} + 0 + \cdots + 0 + \cdots = \frac{3}{2}\,.$$

Hence, the iterated integrals do not coincide also in this case. Therefore, Fubini's theorem can not be applied and since $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is σ-finite the only possibility is that $f \notin L^1(\mu\otimes\mu)$ (as it can be easily verified).