uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

INTEGRATION AND MEASURE | Intermediate Control 1

Time: **90 minutes**

Problem 1 (4 points) Let $(X, \mathcal{A}, \mu)$ be a measure space.

1) Give the definition of σ-algebra and measure.

Prove that:

2) If $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\bigcap_{j=1}^{\infty} A_j \in \mathcal{A}$.

3) If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$.

4) If $A, B \in \mathcal{A}$, $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and, if $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

5) If $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Problem 2 (3 points) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{R}$ be an integrable function.

a) Prove Markov's inequality:

$$\mu(\{x \in X : \ |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f| \, d\mu \, .$$

b) Using Markov's inequality, show that if f is a measurable function, then

$$\begin{array}{llcl} \text{b1)} & \int |f| \, d\mu \ = 0 & \Longleftrightarrow & \mu(f \neq 0) = 0 \, , \\ \text{b2)} & \int |f| \, d\mu \ < \infty & \Longrightarrow & \mu(|f| = \infty) = 0 \, . \end{array}$$

Problem 3 (3 points) Prove that $\displaystyle\lim_{n \to \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = 1$

Hint: Prove that $\left(1 + \dfrac{x}{n}\right)^n \leq e^x$ for $x > 0$ and use an adequate convergence theorem.
