uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

INTEGRATION AND MEASURE

EVALUATION TEST 3

Time: **3 hours**
Remark: You must justify all the steps.

Problem 1 (2,5 points)

a) What type of measure is a Borel-Stieltjes measure? What is a distribution function?

b) Let us consider the function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ x & \text{if} \quad 1 \leq x < 3 \\ 4 & \text{if} \quad x \geq 3 \end{cases}$$

Let μ_F be the Borel-Stieltjes measure with distribution function F. Calculate:

$$\mu_F(\{1\}), \quad \mu_F(\{2\}), \quad \mu_F((1,3]), \quad \mu_F((1,3)), \quad \mu_F([1,3]).$$

c) Give an example of a distribution function F such that

$$\mu_F((a,b)) < F(b) - F(a) < \mu_F([a,b]), \qquad \text{for some } a \text{ and } b.$$

Problem 2 (2,5 points)

a) Prove that the sequence of functions

$$f_n(t) = \left(1 + \frac{t}{n}\right)^n, \qquad t \geq 0,$$

verify that $f_3(t) \leq f_n(t)$ for $n \geq 3$.

b) Calculate

$$\lim_{n\to\infty} \int_0^1 \frac{n + n^2 x}{(1+x)^n}\, dx.$$

State correctly the results and theorems you need to arrive at the solution.

Hint for b): To start, do the change of variable $t = nx$.

Problem 3 (2,5 points)

a) State Fubini's theorem for general functions.

b) Prove that the function $f(x,y) = e^{-y}\sin 2xy$ is integrable in $A = [0,1]\times[0,\infty)$.

c) Prove that

$$\int_0^1 e^{-y}\sin 2xy\,dx = \frac{e^{-y}}{y}\sin^2 y\,, \qquad \int_0^\infty e^{-y}\sin 2xy\,dy = \frac{2x}{1+4x^2}\,.$$

d) Using Fubini's theorem, prove that:

$$\int_0^\infty e^{-y}\,\frac{\sin^2 y}{y}\,dy = \frac{1}{4}\log 5\,.$$

Problem 4 (2,5 points)

a) Calculate the Laplace transform of the function $f(t) = t^2\sin(at)$. Indicate what properties you are using.

b) Solve the following initial value problems, obtaining first the Laplace transform $Y(z)$ of the solution $y(t)$ and then, anti-transforming $Y(z)$:

$$\begin{cases} y'' - y' - 2y = (t^3 + t^2)\,e^{2t}\,, & \text{for } t > 0\,,\\ y(0) = 2\,,\ y'(0) = 4\,. \end{cases}$$

You can use the following properties and formulas for the Laplace transform:

PROPERTIES OF LAPLACE TRANSFORM. Let $f, g \in \mathcal{E}$, $\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{R}$.

(1) $\mathcal{L}[\alpha f + \beta g](z) = \alpha\,\mathcal{L}[f](z) + \beta\,\mathcal{L}[g](z)$.

(2) $\mathcal{L}[e^{at} f(t)](z) = \mathcal{L}[f](z-a)$.

(3) $\mathcal{L}[f(at)](z) = \dfrac{1}{a}\,\mathcal{L}[f(t)](z/a), \quad (a > 0)$.

(4) $\mathcal{L}[f(t-a)H(t-a)](z) = e^{-az}L[f](z)$, where $a > 0$ and $H(t) = \begin{cases} 0\,, & \text{if } x \le 0\,, \\ 1\,, & \text{if } x > 0\,, \end{cases}$ (Heaviside function).

(5) If f is continuous and f' is piecewise continuous on $(0, \infty)$, and $f, f' \in \mathcal{E}$, then

$$\mathcal{L}f'(z) = z\,\mathcal{L}f(z) - f(0)\,.$$

(6) If $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $(0, \infty)$, and $f, f', \ldots, f^{(n)} \in \mathcal{E}$, then

$$\mathcal{L}[f^{(n)}](z) = z^n\,\mathcal{L}f(z) - z^{n-1}f(0) - z^{n-2}f'(0) - \cdots - zf^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

(7) $\mathcal{L}f(z)$ is infinitely derivable on $\operatorname{Re} z > \alpha$ if $f(t), t^n f(t) \in \mathcal{E}$ (both with exponential growth α), and

$$\frac{d^n}{dz^n}[\mathcal{L}f(z)] = (-1)^n\,\mathcal{L}[t^n f(t)](z)\,.$$

(8) If $f \in \mathcal{E}$ then $g(t) = \int_0^t f(x)\,dx \in \mathcal{E}$ and $\mathcal{L}g(z) = \frac{1}{z}\,\mathcal{L}f(z)$.

(9) If $f(t)/t$ is integrable on $[0, T]$ for all $T > 0$, then $\mathcal{L}[\frac{f(t)}{t}](z) = \int_z^\infty \mathcal{L}f(z)\,dz$.

LAPLACE TRANSFORMS TABLE

$$f(t) = 1\,, \qquad \mathcal{L}[f](s) = \frac{1}{s} \qquad (s > 0)\,,$$

$$f(t) = t^n \quad (n \in \mathbb{N})\,, \qquad \mathcal{L}[f](s) = \frac{n!}{s^{n+1}} \quad (s > 0)\,,$$

$$f(t) = t^a \quad (a > -1)\,, \qquad \mathcal{L}[f](s) = \frac{\Gamma(a+1)}{s^{a+1}} \quad (s > 0)\ ,$$

$$f(t) = \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{a}{s^2 + a^2} \quad (s > 0)\ ,$$

$$f(t) = \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{s}{s^2 + a^2} \quad (s > 0)\ ,$$

$$f(t) = \frac{\sin at}{t} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \arctan \frac{a}{s} \quad (s > 0)\,,$$

$$f(t) = e^{at} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{1}{s-a} \quad (s > a)\ ,$$

$$f(t) = e^{at} t^b \quad (a \in \mathbb{R},\, b > -1)\,, \qquad \mathcal{L}[f](s) = \frac{\Gamma(b+1)}{(s-a)^{b+1}} \quad (s > a)\,,$$

$$f(t) = e^{at} \sin bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{b}{(s-a)^2 + b^2} \quad (s > a)\ ,$$

$$f(t) = e^{at} \cos bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{s-a}{(s-a)^2 + b^2} \quad (s > a)\ ,$$

$$f(t) = \sinh at = \frac{e^{at} - e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{a}{s^2 - a^2} \quad (s > |a|)\ ,$$

$$f(t) = \cosh at = \frac{e^{at} + e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{s}{s^2 - a^2} \quad (s > |a|)\ ,$$

$$f(t) = \delta_a \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = e^{-as} \quad (s > 0)\,,$$

$$f(t) = \sin at - at \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{2a^3}{(s^2 + a^2)^2}\,,$$

$$f(t) = t\ \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](s) = \frac{2as}{(s^2 + a^2)^2}\,.$$