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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 1: Measure theory

Section 1.2: Measure spaces

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1 Measure Theory

1.2. Measure spaces

Problem 1.2.1 Let X be a set and $\mathcal{A} = \mathcal{P}(X)$. Let us also consider a function $p : X \longrightarrow [0, \infty]$. Now, we define for $A \subseteq X$ the set function

$$\mu(A) := \sum_{x \in A} p(x) = \sup_{\{x_1,\dots,x_n\} \subseteq A} \sum_{j=1}^{n} p(x_j)\,.$$

Prove that μ is a measure on X. In the particular case that $p(x) = 1$ for all $x \in X$, this measure is known as the *counting measure* in X, since in this case $\mu(A) = \sum_{x \in A} 1 = \#A$, the number of elements of A.

Solution: First $\mu(\varnothing) = 0$ since for $\varnothing$ the sum is empty. Now, if $\{A_j\}_{j=1}^{\infty}$ is a sequence of disjoint subsets of X, and $B = \cup_{j=1}^{\infty} A_j$ then, if $x \in B$, x only belongs to a unique A_j. Also, if E, F are contained in $[0, \infty)$ it is easy to check that $\sup(A + B) = \sup A + \sup B$. Hence

$$\mu(B) = \sum_{x \in B} p(x) = \sum_{j=1}^{\infty} \sum_{x \in A_j} p(x) = \sum_{j=1}^{\infty} \mu(A_j)\,.$$

Problem 1.2.2 Let $(X, \mathcal{A})$ be a measurable space and define the function $\delta_{x_0} : \mathcal{A} \longrightarrow [0, \infty]$ by

$$\delta_{x_0}(A) = \begin{cases} 1\,, & \text{if } x_0 \in A\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Prove that δ_{x_0} is a measure on $(X, \mathcal{A})$ (it is called the δ-Dirac measure concentrated at x_0).

Solution: δ_{x_0} is a particular case of the measure defined in problem 1.2.1. Here $p(x) = 1$ if $x = x_0$ and $p(x) = 0$ otherwise.

Problem 1.2.3 Let $(X, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \longrightarrow [0, \infty]$ be a countably additive function on the σ-algebra $\mathcal{A}$.

a) Show that if μ satisfies that $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\varnothing) = 0$ (and therefore μ is a measure).

b) Find an example for which $\mu(\varnothing) \neq 0$ (and therefore the countably subadditivity property does not imply that μ is a measure).

Hint: b) Take $\mu(A) = \infty$ for any set A.
Solution: a) If $\mu(A) < \infty$, then $\mu(A) = \mu(A \cup \varnothing) = \mu(A) + \mu(\varnothing)$ and so $\mu(\varnothing) = 0$. b) It is easy to check that μ defined as $\mu(A) = \infty$ for any set A is countably additive: let $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{A}$ be a sequence of disjoint measurable sets and let $B = \cup_{j=1}^{\infty} A_j$. Then $\mu(B) = \infty = \sum_j \mu(A_j)$.

Problem 1.2.4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)\,.$$

Solution: If $\mu(E) = \infty$ or $\mu(F) = \infty$ then $\mu(E \cup F) = \infty$ and the equality trivially holds. If $\mu(E) < \infty$ and $\mu(F) < \infty$, then $\mu(E \cap F) < \infty$ and so, as $E = (E \setminus F) \cup (E \cap F)$ and this union is disjoint, we deduce that $\mu(E) = \mu(E \setminus F) + \mu(E \cap F) \Rightarrow \mu(E \setminus F) = \mu(E) - \mu(E \cap F)$. Similarly, $\mu(F \setminus E) = \mu(F) - \mu(E \cap F)$. Also, $E \cup F = (E \setminus F) \cup (E \cap F) \cup (F \setminus E)$ and this union is also disjoint. So,

$$\begin{aligned}\mu(E \cup F) &= \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E)\\ &= \mu(E) - \mu(E \cap F) + \mu(E \cap F) + \mu(F) - \mu(E \cap F) = \mu(E) + \mu(F) - \mu(E \cap F).\end{aligned}$$

Problem 1.2.5 Let $(X, \mathcal{M}, \mu)$ be a measure space. Given $E \in \mathcal{M}$ we define

$$\mu_E(A) = \mu(A \cap E)\,, \qquad \text{for all } A \in X\,.$$

Prove that que μ_E is also a measure on $(X, \mathcal{M})$. We say that μ_E is *concentrated* at E because $\mu_E(A) = 0$ when $A \subseteq E^c$.

Solution: a) Since μ is a measure: $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$;
b) Let $\{A_j\}_{j=1}^{\infty}$ be a disjoint countable collection for sets in $\mathcal{A}$. Then the collection $\{A_j \cap E\}_{j=1}^{\infty}$ is also disjoint and, as μ is a measure,

$$\mu_E\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu\Big(\big(\bigcup_{j=1}^{\infty} A_j\big) \cap E\Big) = \mu\Big(\bigcup_{j=1}^{\infty}(A_j \cap E)\Big) = \sum_{j=1}^{\infty} \mu(A_j \cap E) = \sum_{j=1}^{\infty} \mu_E(A_j)\,.$$

Problem 1.2.6 Let X be an infinite countable set. Let us consider the σ-algebra $\mathcal{M} = \mathcal{P}(X)$ and let us define for $A \in \mathcal{M}$:

$$\mu(A) = \begin{cases} 0\,, & \text{if } A \text{ is finite,}\\ \infty\,, & \text{if } A \text{ is infinite.}\end{cases}$$

a) Prove that μ is finitely additive, but not countably additive.

b) Prove that $X = \lim_{n\to\infty} A_n$, being $\{A_n\}_{n=1}^{\infty}$ an increasing sequence of sets such that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$.

Solution: a) μ is finitely additive: Let $A_1, \ldots, A_n$ be disjoint measurable sets. If of all them are finite them $\cup_j A_j$ is also finite and so $\mu(\cup_j A_j) = 0 = \sum_j \mu(A_j)$. If $\exists A_j$ infinite then $\cup_j A_j$ is also infinite ad so $\mu(\cup_j A_j) = \infty = \sum_j \mu(A_j)$. On the other hand, if $X = \{x_n\}_{n=1}^{\infty}$ then $X = \cup_{j=1}^{\infty}\{x_j\}$ and $\mu(X) = \infty \neq \sum_{j=1}^{\infty} \mu(\{x_j\}) = \sum_{j=1}^{\infty} 0 = 0$. Hence μ is not countably additive.
b) If $X = \{x_n\}_{n=1}^{\infty}$ take $A_n = \{x_1, \ldots, x_n\}$. Then $X = \cup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ and $\mu(A_n) = 0$ for all n.

Problem 1.2.7 Let $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ and μ be the counting measure on X. Construct a decreasing sequence of subsets $A_n \in \mathcal{P}(\mathbb{N})$ such that $\cap_n A_n = \varnothing$, but $\lim_{n\to\infty} \mu(A_n) \neq 0$.

Solution: Take $A_n = \{n, n+1, \ldots\}$. Then $A_{n+1} \subset A_n$, $\mu(A_n) = \infty$ for all n, $\cap_{n=1}^{\infty} A_n = \varnothing$ and so $\lim_{n\to\infty} \mu(A_n) = \infty \neq 0$.

Problem 1.2.8* Let $(X, \mathcal{A})$ be a measurable space. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on $(X, \mathcal{A})$.

a) Prove that if $\{\mu_n\}_{n=1}^{\infty}$ is increasing, that is to say that

$$\mu_n(A) \le \mu_{n+1}(A)\,, \qquad \forall A \in \mathcal{A},$$

then

$$\mu(A) := \lim_{n\to\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

b) Prove that for any sequence of measures $\{\mu_n\}_{n=1}^{\infty}$

$$\mu(A) = \sum_{n=1}^{\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

Hints: a) Consider a countable disjoint family $\{A_j\} \subset \mathcal{A}$ and let $A = \cup_j A_j$. If $\mu(A) = \infty$, then for all $M \in \mathbb{N}$, $\exists N = N(M)$ such that $\mu_n(A) > M$ for all $n \geq N$. Prove that then $\exists K \in \mathbb{N}$ such that $\sum_{j=1}^{K} \mu(A_j) > M - 1$. If $\mu(A) < \infty$, then $\mu_n(A) = \sum_{j=1}^{\infty} \mu_n(A_j) \leq \sum_{j=1}^{\infty} \mu(A_j)$ and so, $\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)$. Also, $\mu_n(A) = \sum_{j=1}^{\infty} \mu_n(A_j) \geq \sum_{j=1}^{K} \mu_n(A_j)$ and so, $\mu(A) \geq \sum_{j=1}^{K} \mu(A_j)$ for every K. Hence, $\mu(A) \geq \sum_{j=1}^{\infty} \mu(A_j)$. b) Take $\nu_n = \sum_{j=1}^{n} \mu_j$ and apply a).

Problem 1.2.9 Let $(X, \mathcal{M}, \mu)$ be a measure space such that for all $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$. A measure space or a measure with this property is called *semifinite*.

a) Show that a σ-finite measure is semifinite.

b) Let X be a non countable set. Let $\mathcal{M} = \mathcal{P}(X)$. Let μ be the counting measure. Prove that μ is semifinite but it is not σ-finite.

Solution: a) That X is σ-finite means that $X = \cup_{n=1}^{\infty} X_n$ with X_n disjoint subsets and $\mu(X_n) < \infty$. Let $E \in \mathcal{M}$ with $\mu(E) = \infty$. We have that $E = E \cap X = \cup_{n=1}^{\infty} E_n$ with $E_n = E \cap X_n$ and $\mu(E_{n_0}) > 0$ for some n_0, since on the contrary if $\mu(E_n) = 0$ for all n, then $\mu(E) = \sum_n \mu(E_n) = 0$, a contradiction. As $0 < \mu(E_{n_0}) \leq \mu(X_{n_0}) < \infty$ we see that we can choose $F = E_{n_0}$ and so, X is semifinite.
b) μ can not be σ-finite because if $X = \cup_{n=1}^{\infty} X_n$ with X_n disjoint subsets and $\mu(X_n) < \infty$ then, as μ is the counting measure, the subsets X_n must be finite. But then, X should be a countable union of finite sets, and so X should be countable, a contradiction. Finally, μ is semifinite because if E is a subset with $\mu(E) = \infty$ then $E \neq \varnothing$ and so $\exists x_0 \in E$ and by taking $F = \{x_0\}$ we have $F \subset E$ and $\mu(E) = 1 < \infty$.

Problem 1.2.10 Let $(X, \mathcal{M}, \mu)$ be a semifinite measure space and let $E \in \mathcal{M}$ be a set with $\mu(E) = \infty$.

a) Prove that
$$\sup\{\mu(F) : \; F \in \mathcal{M}, F \subset E, \mu(F) < \infty\} = \infty\,.$$

b) Prove that if c is a positive real number, then there exists a set $F \subset E$ such that $F \in \mathcal{M}$ and $c < \mu(F) < \infty$.

Hint: a) Denote by s the supremum and suppose that $s < \infty$. Show that there exists $F \subset E$ with $\mu(F) = s$. But then if $E' = E \setminus F$ then $\mu(E') = \infty$ and $\exists F' \subset E'$ with $0 < \mu(E') < \infty$. Get a contradiction with the set $F \cup F'$.

Solution: a) Let us suppose that $s < \infty$. Then, from the definition of a supremum, we obtain that there exists a sequence $\{F_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $F_n \subset E$ and $\mu(F_n) \nearrow s$ as $n \to \infty$. The sequence F_n can be even chosen increasing by substituting, if necessary, F_n by $F_n' = F_1 \cup \cdots \cup F_n$. Hence we have found a subset $F = \cup_{n=1}^{\infty} F_n \subset E$ with $\mu(F) = \lim_{n\to\infty} \mu(F_n) = s < \infty$ and so $E \subsetneq F$. Since $\mu(E) = \infty$ the subset $E' = E \setminus F$ must have $\mu(E') = \infty$. As μ is semifinite this implies

that $\exists F' \subset E'$ with $0 < \mu(F') < \infty$. But then $F \cup F' \subset E$ and $\mu(F \cup F') = \mu(F) + \mu(F') = s + \mu(F') > s$, a contradiction with the definition of s. Hence, $s = \infty$.
b) If $c > 0$, then $\exists F \in \mathcal{M}$, $F \subset E$ with $c < \mu(F) < \infty$ because if there not exists such an F, then $s \leq c$ which is impossible since $s = \infty$.

Problem 1.2.11 Let $\{A_n\}$ be measurable sets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that x belongs to only a finite number of A_n's for a.e. $x \in X$. Alternatively, the set A of points x belonging to an infinite number of A_n's, has zero measure (Borel-Cantelli Lemma).

Hint: $A = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$.

Solution: Let $A = \{x \in X : x \in A_n \text{ for infinitely many } n\text{'s}\}$. Then, $x \in A \iff x \in \bigcup_{n=N}^{\infty} A_n$ for all $N \in \mathbb{N}$ and so $A = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$. But then, for all $N \in \mathbb{N}$,

$$\mu(A) \leq \mu\Big(\bigcup_{n=N}^{\infty} A_n\Big) \leq \sum_{n=N}^{\infty} \mu(A_n) \to 0 \qquad \text{as } N \to \infty,$$

since $\sum_{n=N}^{\infty} \mu(A_n)$ are the tails of the convergent series $\sum_{n=1}^{\infty} \mu(A_n)$. Therefore $\mu(A) \leq 0$ and so, as μ is a measure, $\mu(A) = 0$.

Problem 1.2.12* Let $(X, \mathcal{A}, \mu)$ be a measure space, and let

$$\mathcal{N} = \{N \subseteq X : \; N \subseteq B \in \mathcal{A},\, \mu(B) = 0\}.$$

Prove that

i) $\overline{\mathcal{A}} = \{A \cup N : \; A \in \mathcal{A},\, N \in \mathcal{N}\}$ is a σ-algebra. In fact, $\overline{\mathcal{A}}$ is the σ-algebra generated by $\mathcal{A} \cup \mathcal{N}$.

ii) $\overline{\mu} : \overline{\mathcal{A}} \longrightarrow [0, \infty]$ given by $\overline{\mu}(A \cup N) = \mu(A)$ is a well-defined measure and extends μ.

iii) $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space.

Solution: i) a) $\varnothing \in \overline{\mathcal{A}}$ since $\varnothing \in \mathcal{N}$, $\varnothing = \varnothing \cup \varnothing$ and $\varnothing \in \mathcal{A} \cap \mathcal{N}$. b) Let $E \in \mathcal{A}$, $E = A \cup N$ with $A \in \mathcal{A}$, $N \subseteq B$ and $\mu(B) = 0$. Let $M = B \setminus N$. Then $B = M \cup N$ and $E^c = A^c \cap N^c = (A^c \setminus B) \cap (A^c \cap M) \in \overline{\mathcal{A}}$ because $A^c \setminus B \in \mathcal{A}$ and $A^c \cap M \in \mathcal{N}$. c) Let $\{E_j\}_{j=1}^{\infty} \subset \overline{\mathcal{A}}$, $E_j = A_j \cup N_j$, $A_j \in \mathcal{A}$, $N_j \subseteq B_j$, $\mu(B_j) = 0$. Then

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (A_j \cup N_j) = \Big(\bigcup_{j=1}^{\infty} A_j\Big) \bigcup \Big(\bigcup_{j=1}^{\infty} N_j\Big) \in \overline{\mathcal{A}}$$

since $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ and $\bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$ because $\bigcup_{j=1}^{\infty} N_j \subset \bigcup_{j=1}^{\infty} B_j$, $\mu\big(\bigcup_{j=1}^{\infty} B_j\big) \leq \sum_{j=1}^{\infty} \mu(B_j) = 0$.
d) $\overline{\mathcal{A}} = \sigma(\mathcal{A} \cup \mathcal{N})$ since clearly $\mathcal{A} \cup \mathcal{N} \subset \overline{\mathcal{A}}$ and if $\mathcal{B}$ is any σ-algebra containing $\mathcal{A} \cup \mathcal{N}$ then $\forall A \in \mathcal{A}$, $\forall N \in \mathcal{N}$, in particular, $A, N \in \mathcal{B}$ and so $A \cup N \in \mathcal{B}$.

ii) a) $\overline{\mu}$ is well-defined: Let us suppose that $E = A \cup N = A' \cup N' \in \overline{\mathcal{A}}$ with $A, A' \in \mathcal{A}$, $N \subseteq B$, $N' \subseteq B'$, $\mu(B) = \mu(B') = 0$. Then: $A \subseteq A' \cup B' \Rightarrow \mu(A) \leq \mu(A') + \mu(B') = \mu(A')$ and $A' \subseteq A \cup B \Rightarrow \mu(A') \leq \mu(A) + \mu(B) = \mu(A)$. Hence $\mu(A) = \mu(A')$. b) $\overline{\mu}$ is a measure:
- $\varnothing = \varnothing \cup \varnothing \Rightarrow \overline{\mu}(\varnothing) = \mu(\varnothing) = 0$.
- Let $\{E_j\}_{j=1}^{\infty} \subset \overline{\mathcal{A}}$, $E_j = A_j \cup N_j$, $A_j \in \mathcal{A}$, $N_j \in \mathcal{N}$. Then

$$E := \bigcup_{j=1}^{\infty} E_j = \Big(\bigcup_{j=1}^{\infty} A_j\Big) \bigcup \Big(\bigcup_{j=1}^{\infty} N_j\Big) \qquad \text{with } \bigcup_{j=1}^{\infty} A_j \in \mathcal{A},\, \bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$$

and so $\overline{\mu}(E) = \mu\big(\bigcup_{j=1}^{\infty} A_j\big) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \overline{\mu}(E_j)\,.$

iii) Let $F \subseteq E \in \overline{\mathcal{A}}$ with $\overline{\mu}(E) = 0$. Let $E = A \cup N$ with $A \in \mathcal{A}$, $N \in \mathcal{N}$, $N \subseteq B$ and $\mu(B) = 0$. Then $\mu(A) = \overline{\mu}(E) = 0$ and so $E \subseteq A \cup B$, $\mu(A \cup B) \leq \mu(A) + \mu(B) = 0$. Hence as $F \subseteq E \subseteq A \cup B$ and $\mu(A \cup B) = 0$ we conclude that $F \in \mathcal{N} \subset \overline{\mathcal{A}}$ and $\overline{\mu}(F) = 0$.