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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 1: Measure theory

Section 1.3: Construction of measures

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1 Measure Theory

1.3. Construction of measures

Problem 1.3.1 Let μ^* be an outer measure in X and let H be a μ^*-measurable set. Let us consider the restriction μ_0^* of μ^* to $\mathcal{P}(H)$: $\mu_0^*(A) = \mu^*(A \cap H)$ for all $A \subset X$.

i) Check that μ_0^* is an outer measure on H.

ii) Check that $M \subseteq H$ is μ_0^*-measurable if and only if it is μ^*-measurable.

Solution: i) a) $\mu_0^*(\varnothing) = \mu^*(\varnothing) = 0$. b) If $A \subseteq B$ then $A \cap H \subseteq B \cap H$ and $\mu_0^*(A) = \mu^*(A \cap H) \leq \mu^*(B \cap H) = \mu_0^*(B)$. c) Let $\{A_j\}_{j=1}^{\infty}$ be a collection of subsets of X. Then, as μ^* is an outer measure,

$$\mu_0^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \mu^*\Big(\big(\bigcup_{j=1}^{\infty} A_j\big) \cap H\Big) = \mu^*\Big(\bigcup_{j=1}^{\infty}(A_j \cap H)\Big) \leq \sum_{j=1}^{\infty} \mu^*(A_j \cap H) = \sum_{j=1}^{\infty} \mu_0^*(A_j)\,.$$

ii) Let $\mathcal{M} \subset \mathcal{P}(\mathcal{X})$ be the σ-algebra of μ^*-measurable sets and $\mathcal{M}_0 \subset \mathcal{P}(H)$ be the σ-algebra of μ_0^*-measurable sets.
($\Rightarrow$) Let $M \in \mathcal{M}_0$, $M \subseteq H$ and let $A \subseteq X$. Then $\mu_0^*(A \cap M) + \mu_0^*(A \cap M^c) = \mu_0^*(A)$ and so

$$\mu^*(A \cap M \cap H) + \mu^*(A \cap M^c \cap H) = \mu^*(A \cap H).$$

As $M \subseteq H$, we have that $H \cap M^c = H \setminus M = (X \setminus M) \cap H$ and so

$$\mu^*(A \cap M) + \mu^*(A \cap (X \setminus M) \cap H) = \mu^*(A \cap H) \leq \mu^*(A)\,,$$

since $A \cap H \subseteq A$ and μ^* is an outer measure. But $A \cap (X \setminus M) = A \setminus M$ and so $\mu^*(A \cap M) + \mu^*(A \setminus M) \leq \mu^*(A)$. Since μ^* is an outer measure and $(A \cap M) \cup (A \setminus M) = A$ the opposite inequality is trivial. Hence

$$\mu^*(A \cap M) + \mu^*(A \setminus M) = \mu^*(A)\,, \quad \forall A \subseteq X \quad \Longrightarrow \quad M \in \mathcal{M}\,.$$

($\Leftarrow$) Let $M \in \mathcal{M}$, $M \subseteq H$. If $A \subseteq H$, then $\mu^*(A \cap M) + \mu^*(A \cap (X \setminus M)) = \mu^*(A)$ and so

$$\mu^*(A \cap M \cap H) + \mu^*(A \cap (X \setminus M) \cap H) = \mu^*(A \cap H)\,.$$

But this is equivalent to $\mu_0^*(A \cap M) + \mu_0^*(A \cap (H \setminus M)) = \mu_0^*(A)$, and therefore $M \in \mathcal{M}_0$.

Problem 1.3.2

i) Let X be any set. Let us define $\mu^* : \mathcal{P}(X) :\longrightarrow [0,1]$ by $\mu^*(\varnothing) = 0$, $\mu^*(A) = 1$, if $A \neq \varnothing$, $A \subseteq X$. Check that μ^* is an outer measure and determine the σ-algebra $\mathcal{M}$ of measurable sets.

ii) Do the same if $\mu^*(\varnothing) = 0$, $\mu^*(A) = 1$, if $A \neq \varnothing$, $A \subsetneq X$, $\mu^*(X) = 2$.

Hints: i) If $\varnothing \subsetneq M \subsetneq X$, then the definition of μ^*-measurable set fails with $E = X$. ii) If $\operatorname{card}(X) > 2$ and $\{x, y\} \subset M \subsetneq X$ the definition fails with $E = M^c \cup \{x\}$; if $M = \{x\}$ the definition fails with $E = \{x, y\} \subsetneq X$.

Solution: i) a) By definition $\mu^*(\varnothing) = 0$. b) Let $A \subseteq B$ then, if $A = \varnothing$, we have $0 = \mu^*(A) \leq \mu^*(B)$ and, if $A \neq \varnothing$, we have $\mu^*(A) = 1 = \mu^*(B)$. c) Let $\{A_j\}_{j=1}^{\infty}$ be a collection of subsets of X. Then

$$\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \begin{cases} 0, & \text{if } A_j = \varnothing \ \forall j \\ 1, & \text{otherwise} \end{cases} \leq \sum_{j=1}^{\infty} \mu^*(A_j)\,.$$

Hence μ^* is an outer measure. The σ-algebra of μ^*-measurable sets is

$$\mathcal{M} = \{M \subset X : \ \mu^*(E) \geq \mu^*(E \cap M) + \mu^*(E \setminus M)\,,\ \forall E \subseteq X\}\,.$$

But, taking $E = X$, we have: $1 = \mu^*(X) < 1 + 1 = \mu^*(X \cap M) + \mu^*(X \setminus M)$ if $M \neq \varnothing, X$ and so only $\varnothing$ and X can be μ^*- measurable. Trivially $\varnothing, X \in \mathcal{M}$. Hence $\mathcal{M} = \{\varnothing, X\}$.

ii) Like in i) parts a) and b) are trivial; c) Given a collection $\{A_j\}_{j=1}^{\infty}$ of subsets of X we have

$$\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \begin{cases} 0, & \text{if } A_j = \varnothing \ \forall j \\ 1, & \text{if } \varnothing \subsetneq \cup_{j=1}^{\infty} A_j \subsetneq X \\ 2, & \text{otherwise} \end{cases} \leq \sum_{j=1}^{\infty} \mu^*(A_j)\,.$$

Hence μ^* is again an outer measure.

If $\text{card}\,(X) = 2$, $X = \{x, y\}$ and $M = \{x\}$ or $M = \{y\}$ it is easy to check that $\mu^*(E) \geq \mu^*(E \cap M) + \mu^*(E \cap M^c)$ with the four possibilities $E = \varnothing$, $E = \{x\}$, $E = \{y\}$ or $E = X$. Hence $M \in \mathcal{M}$ and $\mathcal{M} = \mathcal{P}(X)$ in this case.

Now, let us suppose that $\text{card}\,(X) > 2$ and $\varnothing \subsetneq M \subsetneq X$:

• If $M = \{x\}$, taking $E = \{x, y\}$ we have that $E \cap M = \{x\}$, $E \cap M^c = \{y\}$ and so $\mu^*(E \cap M) + \mu^*(E \cap M^c) = 1 + 1 = 2 > 1 = \mu^*(E)$. Hence $M \notin \mathcal{M}$.

• If $\{x, y\} \subseteq M \subsetneq X$, taking $E = M^c \cup \{x\}$ we have, since $E \subsetneq X$ (because $y \notin E$):

$$\mu^*(E \cap M) + \mu^*(E \cap M^c) = \mu^*(\{x\}) + \mu^*(M^c) = 1 + 1 = 2 > 1 = \mu^*(E)\,.$$

Hence $M \notin \mathcal{M}$. Therefore $\mathcal{M} = \{\varnothing, X\}$ in this case.

Problem 1.3.3 Show that a finitely additive outer measure is countably additive.

Hint: $\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{\infty} A_j$ for all n.

Solution: Let $\{A_j\}_{j=1}^{\infty}$ be a collection of disjoint subsets of X and $A = \bigcup_{j=1}^{\infty} A_j$. We must prove that $\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A_j)$. As μ^* is an outer measure, the inequality $\leq$ holds. But, for any $N \in \mathbb{N}$, by monotonicity

$$\mu^*(A) \geq \mu^*\Big(\bigcup_{j=1}^{N} A_j\Big) = \sum_{j=1}^{N} \mu^*(A_j)$$

since μ^* is finitely additive. By letting $N \to \infty$ we obtain the $\geq$ inequality.

Problem 1.3.4* Let μ^* be an outer measure on X and let $\mathcal{M}$ be the collection of μ^*-measurable sets. Prove Caratheodory's theorem following the steps:

a) If $\mu^*(M) = 0$ then $M \in \mathcal{M}$.

b) If $M \in \mathcal{M}$ then also $M^c = X \setminus M \in \mathcal{M}$.

c) If $M, N \in \mathcal{M}$ then $M \cup N$, $M \cap N$, $M \setminus N \in \mathcal{M}$.

d) If $\{M_j\}_{j=1}^{\infty}$ is a sequence of disjoints in $\mathcal{M}$, then prove by induction on n that

$$\mu^*(A \cap (\cup_1^n M_j)) = \sum_{j=1}^{n} \mu^*(A \cap M_j)\,, \qquad \forall A \subset X\,, \forall n \in \mathbb{N}\,.$$

e) If $\{M_j\}_{j=1}^{\infty}$ is a sequence of disjoints in $\mathcal{M}$ and $M := \cup_{n=1}^{\infty} M_j$ then

$$\mu^*(A \cap M) = \sum_{j=1}^{\infty} \mu^*(A \cap M_j)\,, \qquad \forall A \subset X\,.$$

f) If $\{M_j\}_{j=1}^{\infty}$ is a sequence of disjoints in $\mathcal{M}$, them $M := \cup_{n=1}^{\infty} M_j \in \mathcal{M}$.

g) $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a measure.

h) $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Hints: c) $A \cap (M \cup N) = (A \cap M) \cup (A \cap M^c \cap N)$. d) By c) $\cup_{j=1}^n M_j \in \mathcal{M}$ and so $\mu^*(A \cap (\cup_{j=1}^{n+1} M_j)) = \mu^*(A \cap (\cup_{j=1}^{n+1} M_j) \cap (\cup_{j=1}^n M_j)) + \mu^*(A \cap (\cup_{j=1}^{n+1} M_j) \setminus (\cup_{j=1}^n M_j)) = \mu^*(A \cap (\cup_{j=1}^n M_j)) + \mu^*(A \cap M_{n+1})$. e) It is a consequence of a). f) Use that $\cup_{j=1}^n M_j \in \mathcal{M}$ by c), and so $\mu^*(A) = \mu^*(A \cap \cup(\cup_{j=1}^n M_j)) + \mu^*(A \setminus (\cup_{j=1}^n M_j))$; use now parts d) and e). g) If $\{A_j\}$ is any collection of subsets in $\mathcal{M}$, then the sets $M_j = A_j \setminus (A_1 \cup \cdots \cup A_{n-1} \in \mathcal{M}$ are disjoints and $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} M_j$.

Problem 1.3.5* Let $\mathcal{E} \subset \mathcal{P}(X)$ be a semialgebra and let $\mu_0 : \mathcal{E} :\longrightarrow [0, \infty]$ be a countable additive set function.

a) Prove that μ_0 is monotone: If $E, F \in \mathcal{E}$, $E \subseteq F$, then $\mu_0(E) \leq \mu_0(F)$.

b) Prove that μ_0 is countably sub-additive: If $E = \cup_{i=1}^{\infty} E_i$ with $E_i, E \in \mathcal{E}$, then

$$\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(E_i)\,.$$

c) Let us define, for all $A \subseteq X$,

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : \; E_i \in \mathcal{E}\,, \; A \subseteq \cup_{i=1}^{\infty} E_i \right\}.$$

d) Prove that μ^* is an outer measure (and so, by Caratheodory's Theorem, the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra and $\mu = \mu^*|_{\mathcal{A}}$ is a complete measure).

e) Prove that $\mathcal{E} \subseteq \mathcal{A}$ and that μ^* is an extension of μ_0: $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{E}$.

Hints: a) If $E_1 \subset E_2$ then, as $\mathcal{E}$ is semialgebra, $E_2 = E_1 \cup E_1^c = E_1 \cup F_1 \cup \cdots F_n$ with $F_j \in \mathcal{E}$ and disjoint. b) Consider the disjoint sets $D_i := E_i \setminus (E_1 \cup \cdots \cup E_{i-1}) = E_i \cap (\cap_{i=1}^{n-1} E_i^c)$ and observe that, as $\mathcal{E}$ is semialgebra, we have that $E_i^c = F_{i1} \cup \cdots \cup F_{ik(i)}$ with $F_{ij} \in \mathcal{E}$ and disjoint. c) Given $\varepsilon > 0$ and sets $\{A_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} \mu^*(A_i) < \infty$, choose for each i a collection $\{E_{ij}\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} \mu_0(E_{ij}) < \mu^*(A_i) + \varepsilon/2^i$. Then $A := \cup_i A_i \subseteq \cup_i \cup_j E_{ij}$ and $\mu^*(A) \leq \sum_i \mu^*(A_i) + \varepsilon$. e) Given $E \in \mathcal{E}$, $A \subset X$ with $\mu^*(A) < \infty$ and $\varepsilon > 0$ there exists $\{E_i\} \subset \mathcal{E}$ such that $A \subset \cup_i E_i$ and $\sum_i \mu_0(E_i) < \mu^*(A) + \varepsilon$; also $E^c = F_1 \cup \cdots \cup F_n$ with $F_j \in \mathcal{E}$ and disjoint. Hence, $E_i = (E_i \cap E) \cup (E_i \cap F_1) \cup \cdots \cup (E_i \cap F_n)$, a disjoint union of sets.

Problem 1.3.6 A semiopen interval in $\mathbb{R}$ is an interval of type $\varnothing$, $[a,b)$, $(-\infty,b)$, $[a,\infty)$ or $(-\infty,\infty)=\mathbb{R}$. A semiopen interval in $\mathbb{R}^n$ is a set of type $I=I_1\times I_2\times\cdots\times I_n$, where each I_j is a semiopen interval in $\mathbb{R}$. Let $\mathcal{E}$ be the collection of semiopen intervals in $\mathbb{R}^n$. Prove that $\mathcal{E}$ is a semialgebra.

Solution: By definition $\varnothing\in\mathcal{E}$. Secondly, if $I=I_1\times I_2\times\cdots\times I_n\in\mathcal{E}$, $J=J_1\times I_2\times\cdots\times J_n\in\mathcal{E}$, then $I\cap J=(I_1\cap J_1)\times(I_2\cap J_2)\times\cdots\times(I_n\cap J_n)\in\mathcal{E}$ since it is easy to check that the intersection of two semiopen intervals in $\mathbb{R}$ is again a semiopen interval.
Finally: In $\mathbb{R}$ it is easy to check that if $I\in\mathcal{E}$, then $I^c=I'\cup I''$ with I',I'' disjoint semiopen intervals. In $\mathbb{R}^n$ $(n\geq 2)$, if $I=I_1\times\cdots\times I_n\in\mathcal{E}$, then

$$\begin{aligned}I^c &= (I_1^c\times\mathbb{R}\times\cdots\times\mathbb{R})\cup(\mathbb{R}\times I_2^c\times\cdots\times\mathbb{R})\cup\cdots\cup(\mathbb{R}\times\cdots\times\mathbb{R}\times I_n^c)\\ &= ((I_1'\cup I_1'')\times\mathbb{R}\times\cdots\times\mathbb{R})\cup(\mathbb{R}\times(I_2'\cup I_2'')\times\cdots\times\mathbb{R})\cup\cdots\cup(\mathbb{R}\times\cdots\times\mathbb{R}\times(I_n'\cup I_n''))\\ &= (I_1'\times\mathbb{R}\times\cdots\times\mathbb{R})\cup(I_1''\times\mathbb{R}\cdots\times\mathbb{R})\cup\cdots\cup(\mathbb{R}\times\cdots\mathbb{R}\times I_n')\cup(\mathbb{R}\times\cdots\times\mathbb{R}\times I_n'')=\cup_{\alpha=1}^{2n}I^\alpha\,,\end{aligned}$$

where the I^α's are disjoint and $I^\alpha\in\mathcal{E}$, for all α.

Problem 1.3.7 Show that a subset $B\subseteq\mathbb{R}$ is Lebesgue-measurable if and only if

$$m^*(I)=m^*(I\cap B)+m^*(I\cap B^c)\,,$$

for every open interval $I\subseteq\mathbb{R}$.

Hint: Given $E\subset\mathbb{R}$ with $m^*(E)<\infty$ and $\varepsilon>0$, consider a sequence of intervals $\{I_n\}$ such that $E\subset\cup_n I_n$ and $\sum_n m(I_n)<m^*(E)+\varepsilon$ and observe that, as each I_n is Lebesgue-measurable, $m(I_n)=m^*(I_n)=m^*(B\cap I_n)+m^*(B^c\cap I_n)$.
Solution: If B is Lebesgue-measurable then the equality holds for all $E\subset\mathbb{R}$ and so also holds for any interval. Reciprocally, let us suppose that the inequality is true for any interval. We must prove that $m^*(E)\geq m^*(E\cap B)+m^*(E\cap B^c)$ for all $E\subseteq\mathbb{R}$ since the other inequality trivially holds since m^* is an outer measure. We may assume that $m^*(E)<\infty$ since in other case the inequality is obvious. Given $\varepsilon>0$, let $\{I_n\}$ be a sequence of intervals such that $E\subset\cup_n I_n$ and $\sum_n m(I_n)<m^*(E)+\varepsilon$. Then, as the intervals I_n are Lebesgue-measurable, using our hypothesis, and the subadditivity and monotonicity of μ^*:

$$\begin{aligned}m^*(E) &> -\varepsilon+\sum_{n=1}^\infty m(I_n)=-\varepsilon+\sum_{n=1}^\infty m^*(I_n)\\ &= -\varepsilon+\sum_{n=1}^\infty\big(m^*(B\cap I_n)+m^*(B^c\cap I_n)\big)\\ &\geq -\varepsilon+m^*\big(\bigcup_{n=1}^\infty(B\cap I_n)\big)+m^*\big(\bigcup_{n=1}^\infty(B^c\cap I_n)\big)\\ &= -\varepsilon+m^*\big(B\cap\big(\bigcup_{n=1}^\infty I_n\big)\big)+m^*\big(B^c\cap\big(\bigcup_{n=1}^\infty I_n\big)\big)\\ &\geq -\varepsilon+m^*(B\cap E)+m^*(B^c\cap E)\,.\end{aligned}$$

By letting $\varepsilon\to 0^+$ we obtain that $m^*(E)\geq m^*(E\cap B)+m^*(E\cap B^c)$.

Problem 1.3.8*

a) Prove that $(\mathbb{R}^n, \mathcal{M}, m)$ is translations invariant:

$$A \in \mathcal{M}, \quad a \in \mathbb{R}^n \qquad \Longrightarrow \qquad a + A \in \mathcal{M} \quad \text{and} \quad m(a+A) = m(A).$$

b) Let $(\mathbb{R}^n, \mathcal{M}, \mu)$ be a translations invariant measure space with μ a Radon measure ($\mu(K) < \infty$ for each compact set K). Prove that there exists $k \geq 0$ such that $\mu = km$.

Hints: a) Consider the measure $\mu(B) = m(a+B)$ for $B \in \mathcal{B}(\mathbb{R}^n)$ and observe that $m(a+I) = m(I)$ for each semi-interval I. Hence $\mu(I) = m(I)$ for I semi-interval. Apply Caratheodory-Hopf's extension theorem. b) Let $k = \mu([0,1] \times \cdots \times [0,1])$ and prove that $\mu(I) = k\, m(I)$, for each semi-interval $I = [0, r_1/q_1] \times \cdots \times [0, r_n/q_n]$ with $r_i/q_i \in \mathbb{Q}$. Using now an approximation argument conclude that the same is true for any semi-interval in $\mathbb{R}^n$. Finally apply Caratheodory-Hopf's extension theorem.

Problem 1.3.9* Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry for the Euclidean norm. that is to say $\|g(x) - g(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$. It is known that any isometry is a composition of a translation and an orthogonal transformation. Recall that $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if U is linear and $UU^T = I$ where I is the identity matrix.
Prove that given any Lebesgue-measurable set M, then $g(M)$ is also a Lebesgue-measurable set and $m(g(M)) = m(M)$.

Hints: By problem 1 it suffices to prove it for an orthogonal transformation U. As U is an homeomorphism (bijective and continuous with continuous inverse) then U sends Borel sets into Borel sets. Define a measure μ by $\mu(A) = m(U(A))$ for $A \in \mathcal{B}(\mathbb{R}^n)$, where U is orthogonal, and prove that μ is translations invariant. Hence $\mu(A) = k\, m(A)$ for any $A \in \mathcal{B}(\mathbb{R}^n)$ and for some constant k. But, if $B = \{x : \|x\| < 1\}$ then prove that $\mu(B) = m(B)$ and so $k = 1$. Finally, if $M \in \mathcal{M}$ then $M = A \cup N$ with $A \in \mathcal{B}(\mathbb{R}^n)$ and $N \subset C \in \mathcal{B}(\mathbb{R}^n)$, $m(C) = 0$. Hence $U(M) = U(A) \cup U(N)$ with $U(A) \in \mathcal{B}(\mathbb{R}^n)$ and $U(N) \subset U(C) \in \mathcal{B}(\mathbb{R}^n)$, $m(U(C)) = m(C) = 0$.

Problem 1.3.10* Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Prove that given any Lebesgue-measurable set, then $T(M)$ is also a Lebesgue-measurable set and

$$m(T(M)) = |\det T|\, m(M).$$

Hints: If $\det T = 0$ is trivial because in this case $T(\mathbb{R}^n)$ is contained in an $(n-1)$-dimensional hyperplane which has zero n-dimensional Lebesgue measure. If $\det T \neq 0$, then T is bijective and can be decomposed as $T = UDV$ with U, V orthogonal transformations and D a linear transformation whose matrix is diagonal. As orthogonal transformations are isometries, by problem 1.3.9, it suffices to prove it for D. Let $\lambda_1, \ldots, \lambda_n$ be the elements of the diagonal of D. If $I = [a_1, b_1) \times \cdots \times [a_n, b_n)$ is a semi-interval in $\mathbb{R}^n$, then $D(I) = [\lambda_1 a_1, \lambda_1 b_1) \times \cdots \times [\lambda_n a_n, \lambda_n b_n)$ and so $m(D(I)) = \lambda_1 \cdots \lambda_n\, m(I)$. Define the measure $\mu(M) = \frac{1}{\lambda_1 \cdots \lambda_n}\, m(D(M))$. By Caratheodory-Hopf's extension theorem we have that $\mu = m$. Finally, observe that $\det T = \det D = \lambda_1 \cdots \lambda_n$.

Problem 1.3.11* Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Prove that there exists a unique Radon measure $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$ such that

$$\mu([a,b)) = g(b^-) - g(a^-), \qquad \forall\, [a,b) \in \mathcal{E}.$$

Here $g(x_0^-)$ denotes the left limit of g at the point x_0. This measure $\mu = \mu_g$ is called the *Borel-Stieltjes measure with distribution function g.*

Hint: Prove that μ is countably additive on the semi-intervals: if $[a,b) = \cup_{j=1}^{\infty}[a_j,b_j)$ then $g(b^-) - g(a^-) = \sum_{j=1}^{\infty} g(b_j^-) - g(a_j^-)$. Then, apply Caratheodory-Hopf's extension theorem.

Problem 1.3.12 Let $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$ be a Radon measure. Prove that there exists an increasing and left-continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mu = \mu_g$. Besides, g is unique unless by adding constants.

Hint: Define $g(t) = \mu([0,t])$ for $t \geq 0$ and $g(t) = -\mu([t,0))$ for $t < 0$ and apply Caratheodory-Hopf's extension theorem.

Solution: We define $g(t) = \mu([0,t])$ for $t \geq 0$ and $g(t) = -\mu([t,0))$ for $t < 0$. Then, g is increasing:

$$\begin{aligned}
&0 < t_1 < t_2 \implies [0,t_1) \subset [0,t_2) \implies g(t_1) \leq g(t_2)\,,\\
&t_1 < t_2 < 0 \implies [t_1,0) \supset [t_2,0) \implies -\mu([t_1,0)) \leq -\mu([t_2,0)) \implies g(t_1) \leq g(t_2)\,,\\
&t_1 < 0 < t_2 \implies g(t_1) \leq 0 \leq g(t_2)\,.
\end{aligned}$$

Secondly, g is left-continuous: Given $t \in \mathbb{R}$, let $\{s_n\} \subset \mathbb{R}$ with $s_n \nearrow t$. If $t > 0$, then

$$g(t) = \mu([0,t)) = \mu\Big(\bigcup_{n=1}^{\infty}[0,s_n)\Big) = \lim_{n\to\infty} \mu([0,s_n)) = \lim_{n\to\infty} g(s_n)\,,$$

and, if $t \leq 0$, since μ is a Radon measure (and so $\mu([s_n,0)) < \infty$):

$$g(t) = -\mu([t,0)) = -\mu\Big(\bigcap_{n=1}^{\infty}[s_n,0)\Big) = -\lim_{n\to\infty} \mu([s_n,0)) = \lim_{n\to\infty} g(s_n)\,.$$

Finally, let us see that $\mu = \mu_g$:

$$\begin{aligned}
&0 < a < b \implies \mu([a,b)) = \mu([0,b) \setminus [0,a)) = \mu([0,b)) - \mu([0,a)) = g(b) - g(a) = g(b^-) - g(a^-),\\
&a < b < 0 \implies \mu([a,b)) = \mu([a,0) \setminus [b,0)) = \mu([a,0)) - \mu([b,0)) = -g(a) + g(b) = g(b^-) - g(a^-),\\
&a < 0 < b \implies \mu([a,b)) = \mu([a,0) \cup [0,b)) = \mu([a,0)) + \mu([0,b)) = -g(a) + g(b) = g(b^-) - g(a^-).
\end{aligned}$$

Therefore $\mu([a,b)) = \mu_g([a,b))$ for all semiopen interval and so $\mu = \mu_g$ by Caratheodory-Hopf's extension theorem. Finally, let us suppose that $g, h : \mathbb{R} \longrightarrow \mathbb{R}$ are increasing and left continuous and $\mu_g = \mu_h$: Let $c = g(0) - h(0)$, then as $g(t) - g(0) = \mu_g([0,t)) = \mu_h([0,t)) = h(t) - h(0)$, we conclude that $g(t) - h(t) = c$ and therefore, g is unique unless by adding constants.

Problem 1.3.13 Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function and let μ_g be the corresponding Borel-Stieltjes measure with distribution function g. Prove that:

a) $\mu_g(\{x\}) = g(x^+) - g(x^-)$.

b) $\mu_g(\{x\}) = 0$ if and only if g is continuous at x.

c) $\mu_g([a,b]) = g(b^+) - g(a^-)$.

d) $\mu_g((a,b)) = g(b^-) - g(a^+)$.

e) $\mu_g((a,b]) = g(b^+) - g(a^+)$.

f) If $I \subset \mathbb{R}$ is an open interval, then $\mu_g(I) = 0$ if and only if g is constant on I.

Solution: a) $\mu_g(\{x\}) = \mu_g(\cap_n[x, x+\frac{1}{n})) = \lim_{n\to\infty}\mu_g([x, x+\frac{1}{n}))) = \lim_{n\to\infty} g((x+\frac{1}{n})^-) - g(x^-) = \lim_{n\to\infty} g(x+\frac{1}{n}) - g(x^-) = g(x^+) - g(x^-)$.
b) $\mu_g(\{x\}) = 0 \iff g(x^+) = g(x^-) \iff g$ is continuous at x.
c) $\mu_g([a,b]) = \mu_g([a,b)\cup\{b\}) = \mu_g([a,b)) + \mu_g(\{b\}) = g(b^-) - g(a^-) + g(b^+) - g(b^-) = g(b^+) - g(a^-)$.
d) $\mu_g((a,b)) = \mu_g([a,b)\setminus\{a\}) = \mu_g([a,b)) - \mu_g(\{a\}) = g(b^-) - g(a^-) - g(a^+) + g(a^-) = g(b^-) - g(a^+)$.
e) $\mu_g((a,b]) = \mu_g([a,b]\setminus\{a\}) = \mu_g([a,b]) - \mu_g(\{a\}) = g(b^+) - g(a^-) - g(a^+) + g(a^-) = g(b^+) - g(a^+)$.
f) ($\Leftarrow$) If $g(t) = \text{const.}\ \forall t \in I = (a,b)$, then $\mu_g(I) = g(b^-) - g(a^+) = \text{const.} - \text{const.} = 0$.
($\Rightarrow$) If $g(t) \neq \text{const.}$, then $\exists s, t \in I$ with $g(s) < g(t)$ and so $g(a^+) \le g(s) < g(t) \le g(b^-) \Rightarrow \mu_g((a,b)) > 0$.

Problem 1.3.14

a) Let us consider the function

$$F(x) = \begin{cases} 0 & \text{if } \ x < 1 \\ x & \text{if } \ 1 \le x < 3 \\ 4 & \text{if } \ x \ge 3\,. \end{cases}$$

Let μ_F be the Borel-Stieltjes measure with distribution function F. Calculate:

$$\mu_F(\{1\}), \quad \mu_F(\{2\}), \quad \mu_F(\{3\}), \quad \mu_F((1,3]), \quad \mu_F((1,3)), \quad \mu_F([1,3]), \quad \mu_F([1,3)).$$

b) Give an example of a distribution function F such that

$$\mu_F((a,b)) < F(b) - F(a) < \mu_F([a,b])\,, \qquad \text{for some } a \text{ and } b.$$

Solution: a) $\mu_F(\{1\}) = F(1^+) - F(1^-) = 1$, $\mu_F(\{2\}) = F(2^+) - F(2^-) = 0$, $\mu_F(\{3\}) = F(3^+) - F(3^-) = 1$, $\mu_F((1,3]) = F(3^+) - F(1^+) = 3$, $\mu_F((1,3)) = F(3^-) - F(1^+) = 2$, $\mu_F([1,3]) = F(3^+) - F(1^-) = 4$, $\mu_F([1,3)) = F(3^-) - F(1^-) = 3$. b) It holds for the function F in a), since $F(3) - F(1) = 4 - 1 = 3$.

Problem 1.3.15 Let $F(x)$ be the distribution function on $\mathbb{R}$ given by

$$F(x) = \begin{cases} 0 & \text{if } \ x \in (-\infty, -1) \\ 1+x & \text{if } \ x \in [-1, 0) \\ 2+x^2 & \text{if } \ x \in [0, 2) \\ 9 & \text{if } \ x \in [2, \infty)\,. \end{cases}$$

If μ_F is the Borel-Stieltjes measure with distribution function F, calculate the measure μ_F of the following sets: $\{2\}$, $[-1/2, 3)$, $(-1, 0] \cup (1, 2)$, $[0, 1/2) \cup (1, 2]$, $A = \{x \in \mathbb{R} : |x| + 2x^2 > 1\}$.

Solution: $\mu_F(\{2\}) = F(2^+) - F(2^-) = 3$, $\mu_F([-1/2, 3)) = F(3^-) - F(-1/2^-) = 17/2$, $\mu_F((-1,0]\cup(1,2)) = F(0^+) - F(-1^+) + F(2^-) - F(1^+) = 5$, $\mu_F([0,1/2)\cup(1,2]) = F(1/2^-) - F(0^-) + F(2^+) - F(1^+) = 29/4$, $\mu_F(A) = \mu_F((-\infty,-1/2)\cup(1/2,\infty)) = F(-1/2^-) - F(-\infty) + F(+\infty) - F(1/2^+) = 29/4$.

Problem 1.3.16 Let μ be the counting measure on $\mathbb{R}$. Let us fix $A \subset \mathbb{R}$, and let us define $\nu(B) = \mu(B \cap A)$ for all $B \subset \mathbb{R}$.

a) If $A = \{1, 2, 3, \ldots, n, \ldots\}$ is ν a Borel-Stieltjes measure? If the answer is affirmative, find the distribution function.

b) And if $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$?

Solution: a) Yes, since μ is a Radon measure because $\mu([a,b]) = \#\{n \in \mathbb{N} : \ a \leq n \leq b\} < \infty$; $F(x) = 0$ if $x < 0$, $F(x) = [x]$ if $x \geq 0$.
b) No, since μ gives infinite measure to some compact intervals: $\mu([0,\varepsilon]) = \#\{n \in \mathbb{N} : \ 1/n \leq \varepsilon\} = \infty$, for all $\varepsilon > 0$.

Problem 1.3.17 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\Phi : X \longrightarrow Y$ be a mapping. We define the *image space measure* $(Y, \mathcal{B}, \nu)$ as

$$\mathcal{B} = \Phi(\mathcal{A}) := \{B \subseteq Y : \ \Phi^{-1}(B) \in \mathcal{A}\}$$

and $\nu = \Phi(\mu) : \mathcal{B} \longrightarrow [0,\infty]$ given by $\nu(B) = \mu(\Phi^{-1}(B))$ for all $B \in \mathcal{B}$.
Prove that $(Y, \mathcal{B}, \nu)$ is a measure space and it is complete when $(X, \mathcal{A}, \mu)$ is.

Solution: a) $\nu(\varnothing) = \mu(\Phi^{-1}(\varnothing)) = \mu(\varnothing) = 0$.
b) If $\{B_j\}_{j=1}^{\infty} \subset \mathcal{B}$ is a collection of disjoint subsets of Y, then $\{\Phi^{-1}(B_j)\}_{j=1}^{\infty} \subset \mathcal{A}$ is a collection of disjoint subsets of X and as μ is a measure:

$$\nu\Big(\bigcup_{j=1}^{\infty} B_j\Big) = \mu\Big(\Phi^{-1}\big(\bigcup_{j=1}^{\infty} B_j\big)\Big) = \mu\Big(\bigcup_{j=1}^{\infty} \Phi^{-1}(B_j)\Big) = \sum_{j=1}^{\infty} \mu(\Phi^{-1}(B_j)) = \sum_{j=1}^{\infty} \nu(B_j).$$

Finally, if $(X, \mathcal{A}, \mu)$ is complete and $N \subseteq B \in \mathcal{B}$ with $\nu(B) = 0$ then $\Phi^{-1}(N) \subseteq \Phi^{-1}(B) \in \mathcal{A}$ and $\mu(\Phi^{-1}(B)) = \nu(B) = 0$ and so $\Phi^{-1}(N) \in \mathcal{A}$ and $\mu(\Phi^{-1}(N)) = 0$. Hence $N \in \mathcal{B}$ and $\nu(N) = \mu(\Phi^{-1}(N)) = 0$. Hence $(Y, \mathcal{B}, \nu)$ is also complete.

Problem 1.3.18

a) Let $g : I \longrightarrow \mathbb{R}$ be a continuous and strictly increasing function. As g is injective it has a continuous inverse g^{-1}. Prove that $\mu_g = g^{-1}(m)$, that is to say that the Borel-Stieltjes measure with distribution function g coincides with the image measure of Lebesgue measure under g^{-1}.

b) Let $g : (0,\infty) \longrightarrow \mathbb{R}$ be the function $g(t) = \log t$. Prove that $\mu_g = g^{-1}(m) = e^m$ is invariant under dilations.

Hints: a) Prove that both measures coincide for semi-intervals $[a,b)$ and apply Caratheodory-Hopf's extension theorem. b) Use part a) and the fact that Lebesgue measure is translation invariant. Alternatively, it can be also proved by using Caratheodory-Hopf's extension theorem.
Solution: a) Let $[a,b) \in \mathcal{E}$, the semialgebra of semiopen intervals in $\mathbb{R}$. Then, as g is increasing, $g([a,b)) = [g(a), g(b))$ and so,

$$g^{-1}(m)([a,b))) = m(g([a,b))) = m([g(a), g(b))) = g(a) - g(b) = \mu_g([a,b))\,,$$

since g is continuous. By Caratheodory-Hopf's extension theorem we obtain that $\mu_g = g^{-1}(m)$.
b) Let E be a borelian set in $(0,\infty)$ and let $\lambda > 0$. Then, using a) and the fact that Lebesgue measure is translation invariant:

$$\begin{aligned}\mu_g(\lambda E) &= g^{-1}(m)(\lambda E) = m(g(\lambda E)) = m(\log(\lambda E)) \\ &= m(\log\lambda + \log E) = m(\log E) = m(g(E)) = g^{-1}(m)(E) = \mu_g(E).\end{aligned}$$

Problem 1.3.19 Let $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$ be the unit ball of $\mathbb{R}^n$ and $S_{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere. Let us consider the projection $\pi : B_n \setminus \{0\} \longrightarrow S_{n-1}$ given by

$\pi(x) = x/\|x\|$. The $(n-1)$-dimensional Lebesgue measure on S_{n-1} is defined as $\sigma = n \cdot \pi(m)$, that is to say

$$\sigma(U) = n \cdot m(\pi^{-1}(U)), \qquad \text{for all } U \in \mathcal{B}(S_{n-1}).$$

Prove that σ is invariant under rotations.

Hint: Use problem 1.3.9.

Solution: That σ is invariant under rotations means that $\sigma(T(U)) = \sigma(U)$ for any orthogonal transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. But using problem 1.3.9, we have that

$$\sigma(T(U)) = n \cdot m(\pi^{-1}(T(U))) = n \cdot m(T(\pi^{-1}(U))) = n \cdot m(\pi^{-1}(U)) = \sigma(U).$$

Problem 1.3.20 Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let us consider the product set $X \times Y = \{(x, y) : \ x \in X, y \in Y\}$. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra generated by the set $\mathcal{E} = \{A \times B : \ A \in \mathcal{A}, \ B \in \mathcal{B}\}$. Prove that there exists a unique measure $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \longrightarrow [0, \infty]$ such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B), \qquad \text{for all } A \in \mathcal{A}, \ B \in \mathcal{B}.$$

Hint: Prove that $\mathcal{E}$ is a semi-algebra and that $\mu \otimes \nu$ is countably additive on $\mathcal{E}$. Then apply Caratheodory-Hopf's extension theorem.

Solution: Let us check first that $\mathcal{E}$ is a semialgebra: a) $\varnothing = \varnothing \times \varnothing \in \mathcal{E}$. b) Let $A \times B, A' \times B' \in \mathcal{E}$. Then $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ and, since $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras, $A \cap A' \in \mathcal{A}$ and $B \cap B' \in \mathcal{B}$. Hence $(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B') \in \mathcal{E}$. c) Let $A \times B \in \mathcal{E}$. Then $(A \times B)^c = (A^c \times Y) \cup (A \times B^c)$ and $A^c \times Y, A \times B^c \in \mathcal{E}$ (since $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras) and they are disjoint. Let us define now the set-function $\alpha : \mathcal{E} \longrightarrow [0, \infty]$ given by $\alpha(A \times B) = \mu(A)\, \nu(B)$. It is not difficult to check that α is countably additive. Besides, as μ and ν are σ-finite we have that: $X = \cup_n X_n$ with $\mu(X_n) < \infty$, $X_1 \subseteq X_2 \subseteq \cdots$ and $Y = \cup_n Y_n$ with $\nu(Y_n) < \infty$, $Y_1 \subseteq Y_2 \subseteq \cdots$. Hence $X \times Y = \cup_n (X_n \times Y_n)$ with $\alpha(X_n \times Y_n) = \mu(X_n)\nu(Y_n) < \infty$ and so α is also σ-finite. As a consequence of Caratheodory-Hopf's extension theorem we deduce that there exists a unique measure $\mu \otimes \nu$ defined on the product σ-algebra $\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{E})$ such that $\mu \otimes \nu|_{\mathcal{E}} = \alpha$, that is to say such that $(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.