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Departamento de Matemáticas

# Integration and Measure. Problems

## Chapter 2: Integration theory

### Section 2.1: Integration of positive functions

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# 2 Integration Theory

## 2.1 Integration of positive functions

**Problem 2.1.1** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g : X \longrightarrow [0, \infty]$ be measurable positive functions and $A, B, E \in \mathcal{A}$, $\lambda \geq 0$. Prove that:

*i)* $\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu$ .

*ii)* $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

*iii)* $f \leq g \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

*iv)* $A \subseteq B \implies \int_A f \, d\mu \leq \int_B f \, d\mu$.

*v)* $\int_E f = 0 \Leftrightarrow f = 0$ a.e. in $E$.

*vi)* $\mu(E) = 0 \implies \int_E f \, d\mu = 0$.

*vii)* $A \cap B = \varnothing \implies \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$.

*viii)* $\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$.

*ix)* $f \geq g$, $\int_X g \, d\mu < \infty \implies \int_X f \, d\mu - \int g \, d\mu = \int_X (f - g) \, d\mu$.

*x)* $f \leq g$ a.e. in $E \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

*xi)* $f = g$ a.e. in $E \implies \int_E f \, d\mu = \int_E g \, d\mu$.

*Hints: vi)* If $f = 0$ a.e. and $s = \sum_j c_j \chi_{A_j} \leq f$, then $\mu(A_j) = 0$ for all $j$ and so $s = 0$ a.e. On the other hand, if $\mu(f > 0) > 0$, then $\mu(A) > 0$ for some $n \in \mathbb{N}$, where $A = \{f > 1/n\}$. Hence, $0 \leq s = \frac{1}{n} \chi_A \leq f$ and $\frac{1}{n} \mu(A \cap E) \leq \int_E f \, d\mu$, a contradiction. For the other statements, the idea is always to approximate positive functions by simple functions.

*Solution:* i) Let us observe that $s$ is a simple function if and only if $\tilde{s} = s/\lambda$ is also a simple function. Hence,

$$\int_E \lambda f \, d\mu = \sup_{\substack{0 \leq s \leq \lambda f \\ s \text{ simple}}} \int_E s \, d\mu = \sup_{\substack{0 \leq s/\lambda \leq f \\ s \text{ simple}}} \lambda \int_E \frac{s}{\lambda} \, d\mu = \lambda \sup_{\substack{0 \leq \tilde{s} \leq f \\ \tilde{s} \text{ simple}}} \int_E \tilde{s} \, d\mu = \lambda \int_E f \, d\mu \, .$$

ii) We have that

$$\int_E f \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_E s \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s = \sum_j c_j \chi_{A_j}}} \sum_j c_j \, \mu(A_j \cap E) \, .$$

Now, it is easy to check that if the sets $A_j$ are disjoint then

$$s = \sum_j c_j \chi_{A_j} \leq f \qquad \Leftrightarrow \qquad \tilde{s} = \sum_j c_j \chi_{A_j \cap E} \leq f \, \chi_E$$

and so

$$\int_X f \, \chi_E \, d\mu = \sup_{\substack{0 \leq \tilde{s} \leq f \, \chi_E \\ \tilde{s} = \sum_j c_j \chi_{A_j \cap E}}} \int_X \tilde{s} \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s = \sum_j c_j \chi_{A_j}}} \sum_j c_j \, \mu(A_j \cap E) = \int_E f \, d\mu \, .$$

iii) As $f \leq g$ we have that

$$\{s \text{ simple}: \; 0 \leq s \leq f\} \subseteq \{s \text{ simple}: \; 0 \leq s \leq g\}$$

and so

$$\int_E f \, d\mu = \sup_{0 \leq s \leq f} \int_E s \, d\mu \leq \sup_{0 \leq s \leq g} \int_E s \, d\mu = \int_E g \, d\mu \, .$$

iv) $A \subseteq B \implies \chi_A \leq \chi_B \implies f\chi_A \leq f\chi_B$ and so iv) follows from iii).

v) ($\Leftarrow$) If $f = 0$ a.e. in $E$ and $0 \leq s = \sum_j c_j \chi_{A_j} \leq f$, then $\mu(A_j \cap E) = 0$ for all $j$. Hence,

$$\int_E f \, d\mu = \sup_{\substack{0 \leq s \leq f \\ s \text{ simple}}} \int_E s \, d\mu = \sup_{0 \leq s \leq f} \sum_j c_j \mu(A_j \cap E) = 0 \, .$$

($\Rightarrow$) Assume that $\mu(\{x \in E : f(x) > 0\}) > 0$. As $\{x \in E : f(x) > 0\} = \cup_{n=1}^{\infty} \{x \in E : f(x) > \frac{1}{n}\}$ we have that $\exists \, n \in \mathbb{N}$ such that $\mu(\{x \in E : f(x) > \frac{1}{n}\}) > 0$. Let $A := \{x \in E : f(x) > \frac{1}{n}\}$. Hence, $\mu(A \cap E) = \mu(A) > 0$ and $0 \leq s := \frac{1}{n}\chi_A < f$ and so

$$\int_E s \, d\mu = \frac{1}{n} \int_E \chi_A \, d\mu \leq \int_E f \, d\mu \implies \frac{1}{n} \mu(A \cap E) \leq \int_E f \, d\mu \implies \int_E f \, d\mu > 0 \, .$$

vi) If $\mu(E) = 0$, then $f = 0$ a.e. in $E$ and so, by v), $\int_E f \, d\mu = 0$.

vii) If $A \cap B = \varnothing$, then it is easy to check that $\chi_{A \cup B} = \chi_A + \chi_B$ and so

$$\int_{A \cup B} f \, d\mu = \int_X f \, \chi_{A \cup B} \, d\mu = \int_X f \, (\chi_A + \chi_B) \, d\mu = \int_X f \, \chi_A \, d\mu + \int_X f \, \chi_B \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu \, .$$

viii) Let $\{s_n\}$ and $\{t_n\}$ be sequences of simple functions such that $0 \leq s_n \nearrow f$ and $0 \leq t_n \nearrow g$ as $n \to \infty$. Then $\{s_n + t_n\}$ is a sequence of simple functions such that $s_n + t_n \nearrow f + g$ as $n \to \infty$. By the monotone convergence theorem

$$\begin{aligned} \int_X (f+g) \, d\mu &= \lim_{n \to \infty} \int_X (s_n + t_n) \, d\mu = \lim_{n \to \infty} \Big( \int_X s_n \, d\mu + \int_X t_n \, d\mu \Big) \\ &= \lim_{n \to \infty} \int_X s_n \, d\mu + \lim_{n \to \infty} \int_X t_n \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu \, . \end{aligned}$$

Now, using ii) we get that

$$\int_E (f+g) \, d\mu = \int_X (f+g)\chi_E \, d\mu = \int_X f\chi_E \, d\mu + \int_X fg\chi_E \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu \, .$$

ix) As $f \geq g$ we have that $f - g \geq 0$ and, by ii),

$$\int_X f \, d\mu = \int_X (g + (f-g)) \, d\mu = \int_X g \, d\mu + \int_X (f-g) \, d\mu \, .$$

To finish we subtract $\int_X g \, d\mu < \infty$ from both members.

x) If $A := \{x \in E : f(x) > g(x)\}$, then $\mu(A) = 0$. Let $h(x) = f(x)$ if $x \notin A$ and $h(x) = 0$ otherwise (alternatively $h = f\chi_{A^c}$). Then $f = h$ a.e. and $h$ is measurable. Also $f \geq h$ for all $x \in X$. Applying v) to $f - h$ we get that $\int_X (f-h) \, d\mu = 0$ and so, by ix), $\int_X f \, d\mu = \int_X h \, d\mu$. But also $g - h \geq 0$ for all $x \in X$ and so, by iii), $\int_X (f-h) \, d\mu \geq 0 \Rightarrow \int_X h \, d\mu \leq \int_X g \, d\mu$.

xi) As $f = g$ a.e. we have that $f \leq g$ a.e. and $g \leq f$ a.e. and applying x) twice we get xi).

**Problem 2.1.2** Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $X = \cup_n X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of measurable subsets of $X$. Prove that if $f : X \longrightarrow [0, \infty]$ is a measurable positive function, then

$$\int_X f\,d\mu = \sum_n \int_{X_n} f\,d\mu\,.$$

*Hint:* Use the monotone convergence theorem.

*Solution:* By definition of the sum of a series and using property viii) of problem 2.1.1 for finite sums,

$$\sum_{n=1}^{\infty} \int_{X_n} f\,d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{X_n} f\,d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f\chi_{X_n}\,d\mu = \lim_{N\to\infty} \int_X \sum_{n=1}^{N} \left(f\chi_{X_n}\right) d\mu = \int_X f\,d\mu\,,$$

by the monotone convergence theorem, since the sets $X_n$ are pairwise disjoint, we have that $\sum_{n=1}^{\infty} \chi_{X_n} = \chi_{\cup_n X_n} = \chi_X = 1$ and so $\sum_{n=1}^{N} f\chi_{X_n} \nearrow f$ as $N \to \infty$.

**Problem 2.1.3** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow [0, \infty]$ be a measurable positive function. Let us define

$$\varphi(E) = \int_E f\,d\mu\,, \qquad \text{for all } E \in \mathcal{A}.$$

Prove that $\varphi$ is a measure on $\mathcal{A}$ and that

$$\int_X g\,d\varphi = \int_X gf\,d\mu\,, \qquad \text{for all } g : X \longrightarrow [0, \infty] \text{ measurable.} \tag{1}$$

*Note:* This fact justifies the notation $d\varphi = f\,d\mu$.

*Hint:* Apply Exercise 2.1.1 to prove that $\varphi$ is a measure. Then, prove (1) first for characteristic functions and simple functions and then approximate any positive function for a sequence of simple functions.

*Solution:* a) $\varphi(\varnothing) = \int_{\varnothing} f\,d\mu = 0$ since $\mu(\varnothing) = 0$. b) Let $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$ be a collection of disjoint sets and let $A = \cup_{j=1}^{\infty} A_j$. Then $\chi_A = \sum_{j=1}^{\infty} \chi_{A_j}$ and, as $f, \chi_{A_j} \geq 0$, the monotone convergence theorem gives

$$\varphi(A) = \int_A f\,d\mu = \int_X f\,\chi_A\,d\mu = \int_X \Big(f \sum_{j=1}^{\infty} \chi_{A_j}\Big) d\mu = \sum_{j=1}^{\infty} \int_X f\,\chi_{A_j}\,d\mu = \sum_{j=1}^{\infty} \int_{A_j} f\,d\mu = \sum_{j=1}^{\infty} \varphi(A_j)\,.$$

Hence, $\varphi$ is a measure. On the other hand, if $s = \sum_{j=1}^{n} c_j \chi_{E_j}$ is a simple function:

$$\begin{aligned} \int_X s\,d\varphi &= \sum_{j=1}^{n} c_j \int_X \chi_{E_j}\,d\varphi = \sum_{j=1}^{n} c_j \varphi(E_j) = \sum_{j=1}^{n} c_j \int_{E_j} f\,d\mu \\ &= \sum_{j=1}^{n} c_j \int_X f\,\chi_{E_j}\,d\mu = \int_X \Big(\sum_{j=1}^{n} c_j\,\chi_{E_j}\Big) f\,d\mu = \int_X s\,f\,d\mu \end{aligned} \tag{2}$$

and so (1) holds for simple functions. Now, let$g \geq 0$ measurable and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of simple functions such that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \nearrow g$ as $n \to \infty$. Then, by the monotone convergence theorem and (2),

$$\int_X g\,d\varphi = \lim_{n\to\infty} \int_X s_n\,d\varphi = \lim_{n\to\infty} \int_X s_n f\,d\mu = \int_X gf\,d\mu\,.$$

**Problem 2.1.4** Let $f : [0,1] \longrightarrow [0,\infty]$ be defined by $f(x) = 0$ if $x$ is rational, and otherwise $f(x) = n$ where $n$ is the number of zeros immediately after the decimal point in the representation of $x$ in the decimal scale. Calculate $\int f(x)\, dm$, being $m$ the Lebesgue measure.

*Hint:* $f(x) = k$ for $x \in [1/10^{k+1}, 1/10^k) \setminus \mathbb{Q}$.
*Solution:* If $x \in [1/10^{k+1}, 1/10^k) \setminus \mathbb{Q}$, then $f(x) = k$ and so $f = \sum_{k=1}^{\infty} k\, \chi_{[1/10^{k+1}, 1/10^k)}$ a.e. By the monotone convergence theorem

$$\int f\, dm = \sum_{k=1}^{\infty} k \left( \frac{1}{10^k} - \frac{1}{10^{k+1}} \right) = \sum_{k=1}^{\infty} k\, \frac{9}{10^{k+1}} = 9 \sum_{k=1}^{\infty} \frac{k}{10^{k+1}}\,.$$

Let $F(x) = \sum_{k=1}^{\infty} k x^{k-1}$ and $G(x) = \sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $0 \le x < 1$. Then $F(x) = G'(x) = 1/(1-x)^2$ and so

$$\int f\, dm = \frac{9}{10^2}\, F(1/10) = \frac{9}{10^2}\, \frac{1}{(1-\frac{1}{10})^2} = \frac{1}{9}\,.$$

**Problem 2.1.5** Let $f(x)$ be the function defined in $(0,1)$ by $f(x) = 0$ if $x$ is rational, and $f(x) = [1/x]$ if $x$ is irrational ($[t]$ denote the integer part of $t$). Decide whether or not $f$ is Lebesgue integrable and calculate $\int f(x)\, dm$ being $m$ the Lebesgue measure.

*Hint:* $f(x) = k$ for $x \in (1/(k+1), 1/k] \setminus \mathbb{Q}$.
*Solution:* As $[\frac{1}{x}] = k$ if and only if $x \in (1/(k+1), 1/k] \setminus \mathbb{Q}$ we have that $f = \sum_{k=1}^{\infty} k \chi_{(1/(k+1), 1/k]}$. So, by the monotone convergence theorem

$$\int_0^1 f\, dm = \sum_{k=1}^{\infty} k \left( \frac{1}{k} - \frac{1}{k+1} \right) = \sum_{k=1}^{\infty} k\, \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty\,.$$

Hence, $f$ is not Lebesgue integrable in $(0,1)$.

**Problem 2.1.6** Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e. $\mu(X) = 1$. Let $E \in \mathcal{A}$ be a set with $0 < \mu(E) < 1$. Put $f_n = \chi_E$ if $n$ is odd, $f_n = 1 - \chi_E$ if $n$ is even. What is the relevance of this example to Fatou's lemma?

*Hint:* $\liminf_{n\to\infty} f_n(x) = 0$ for all $x \in X$ but $\liminf_{n\to\infty} \int f_n\, d\mu = \min\{\mu(E), 1 - \mu(E)\}$.
*Solution:* We have that $\liminf_{n\to\infty} f_n(x) = \min\{\chi_E(x), 1 - \chi_E(x)\} = 0$ for all $x \in X$, but it is clear that

$$\int_X f_n\, d\mu = \begin{cases} \mu(E), & \text{if } n \text{ is odd} \\ 1 - \mu(E), & \text{if } n \text{ is even} \end{cases} \implies \liminf_{n\to\infty} \int_X f_n\, d\mu = \min\{\mu(E), 1 - \mu(E)\} > 0.$$

Therefore, in this case, Fatou's lemma tell us that $0 \le \min\{\mu(E), 1 - \mu(E)\}$. Hence, the inequality in Fatou's lemma can be strict!

**Problem 2.1.7** Let $f_{2n-1} = \chi_{[0,1]}$, $f_{2n} = \chi_{[1,2]}$, $n = 1, 2, \ldots$. Check that Fatou's Lemma is verified strictly for this sequence of functions.

*Hint:* $\liminf_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R}\setminus\{1\}$ but $\liminf_{n\to\infty} \int f_n\, dm = 1$.
*Solution:* As $[0,1]\cap[1,2] = \{1\}$ we have that $\liminf_{n\to\infty} f_n(x) = \min\{\chi_{[0,1]}(x), \chi_{[1,2]}(x)\} = 0$ for all $x \in \mathbb{R}\setminus\{1\}$. But $\int_{\mathbb{R}} f_n\, dm = 1$ for all $n \in \mathbb{N}$ and so $\liminf_{n\to\infty} \int f_n\, dm = 1$. Therefore, in this case, Fatou's lemma tell us that $0 \le 1$ and so, the inequality in Fatou's lemma is strict.

**Problem 2.1.8**

a) Check that $\displaystyle\int_1^\infty \frac{1}{x}\, dm = \infty$, being $m$ the Lebesgue measure.

b) Let $p \in \mathbb{R}$. Prove that:

b1) $\displaystyle\int_0^\infty e^{-px} dm < \infty$ if and only if $p > 0$.

b2) $\displaystyle\int_1^\infty \frac{1}{x^p}\, dm < \infty$ if and only if $p > 1$.

b3) $\displaystyle\int_0^1 \frac{1}{x^p}\, dm < \infty$ if and only if $p < 1$.

*Hint:* a) $\displaystyle\frac{1}{x} = \lim_{N\to\infty} \frac{1}{x}\chi_{[1,N]}(x)$. Apply the monotone convergence theorem.
*Solution:* a) As $\frac{1}{x}\chi_{[1,N]}(x) \nearrow \frac{1}{x}\chi_{[1,\infty)}(x) = \frac{1}{x}$, by monotone convergence theorem,

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{N\to\infty} \int_1^\infty \frac{1}{x}\chi_{[1,N]}(x)\, dx = \lim_{N\to\infty} \int_1^N \frac{1}{x}\, dx\,.$$

But $\frac{1}{x}$ is continuous in the bounded interval $[1,N]$ and so it is Riemann-integrable in $[1,N]$ and its Lebesgue integral coincide with its Riemann integral, and to compute it we can use Barrow's rule. Hence,

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{N\to\infty} [\log x]_{x=1}^{x=N} = \lim_{N\to\infty} \log N = \infty.$$

b.1) Similarly,

$$\int_0^\infty e^{-px} dx = \lim_{N\to\infty} \int_0^\infty e^{-px}\chi_{[0,N]}(x)\, dx = \lim_{N\to\infty} \int_0^N e^{-px} dx = \lim_{N\to\infty} \left[\frac{e^{-px}}{-p}\right]_{x=0}^{x=N} = \lim_{N\to\infty} \frac{1}{p}(1-e^{-pN}).$$

Hence, $\int_0^\infty e^{-px} dx < \infty \iff p > 0$ and in this case $\int_0^\infty e^{-px} dx = 1/p$.
b.2) Similarly, if $p \neq 1$,

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\, dx &= \lim_{N\to\infty} \int_1^\infty \frac{1}{x^p}\chi_{[1,N]}(x)\, dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^p}\, dx \\ &= \lim_{N\to\infty} \left[\frac{1}{(1-p)x^{p-1}}\right]_{x=1}^{x=N} = \frac{1}{1-p} \lim_{N\to\infty} \left(\frac{1}{N^{p-1}} - 1\right).\end{aligned}$$

Hence, taking also into account a), $\int_1^\infty \frac{1}{x^p}\, dx < \infty \iff p > 1$ and in this case $\int_1^\infty \frac{1}{x^p}\, dx = \frac{1}{p-1}$.
If $p = 1$, then a similar argument gives that the integral is $\infty$.

b.3) Similarly, if $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{N\to\infty}\int_0^1 \frac{1}{x^p}\,\chi_{[1/N,1]}(x)\,dx = \lim_{N\to\infty}\int_{1/N}^1 \frac{1}{x^p}\,dx$$
$$= \lim_{N\to\infty}\left[\frac{1}{(1-p)x^{p-1}}\right]_{x=1/N}^{x=1} = \frac{1}{1-p}\lim_{N\to\infty}\left(1-N^{p-1}\right)$$

Hence, taking also into account a), $\int_1^\infty \frac{1}{x^p}\,dx < \infty \iff p<1$ and in this case $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}$. If $p=1$, then a similar argument gives that the integral is $\infty$.

**Problem 2.1.9** Prove that the function $f(x) = \frac{1}{\sqrt{x}}$ if $x \in (0,1]$, and $f(0)=0$, is Lebesgue-integrable in $[0,1]$ and calculate its integral.

*Hint:* $f$ is almost everywhere continuous and $f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{x}}\chi_{[\varepsilon,1]}(x)$ if $x\in[0,1]$.

*Solution:* As $\frac{1}{\sqrt{x}}\,\chi_{[1/N,1]}(x) \nearrow \frac{1}{\sqrt{x}}\,\chi_{(0,1]}(x) = f(x)$ when $N\to\infty$ for $x\in[0,1]$, by the monotone convergence theorem,

$$\int_0^1 f(x)\,dx = \lim_{N\to\infty}\int_0^1 f(x)\,\chi_{[1/N,1]}(x)\,dx = \lim_{N\to\infty}\int_{1/N}^1 f(x)\,dx\,.$$

But $f(x)$ is continuous in the bounded interval $[1/N,1]$ and so it is Riemann-integrable in $[1/N,1]$ and its Lebesgue integral coincide with its Riemann integral, and to compute it we can use Barrow's rule. Hence,

$$\int_1^\infty f(x)\,dx = \lim_{N\to\infty}\int_{1/N}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{N\to\infty}[2\sqrt{x}\,]_{x=1/N}^{x=1} = \lim_{N\to\infty} 2\Big(\frac{1}{\sqrt{N}}-1\Big) = 2.$$

**Problem 2.1.10** Let $(X,\mathcal{A},\mu)$ be a measure space and let $f: X \longrightarrow [0,\infty]$ be a measurable positive function. Let $f_n(x) = \min\{f(x), n\}$. Prove that $\displaystyle\int_X f_n\,d\mu \nearrow \int_X f\,d\mu$.

*Hint:* Use an adequate convergence theorem.

*Solution:* It is clear that $f_n(x) \nearrow f(x)$ as $n\to\infty$, by the monotone convergence theorem, as $n\to\infty$. Thus, we have that $\int_X f_n\,d\mu \nearrow \int_X f\,d\mu$ as $n\to\infty$.

**Problem 2.1.11** Let $(X,\mathcal{A},\mu)$ be a measure space and let $f_n : X \longrightarrow [0,\infty]$ be a sequence of measurable positive functions. Let us suppose that $\exists \lim_{n\to\infty} f_n = f$ and that $f_n \le f$ for all $n$. Prove that $\displaystyle\int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu$.

*Hint:* Use Fatou's Lemma and $\int_X f_n\,d\mu \le \int_X f\,d\mu$.

*Solution:* By Fatou's lema

$$\int_X f\,d\mu = \int_X \lim_{n\to\infty} f\,d\mu = \int_X \liminf_{n\to\infty} f\,d\mu \le \liminf_{n\to\infty}\int_X f_n\,d\mu\,.$$

Also, as $f_n \le f$ we have that $\int_X f_n\,d\mu \le \int_X f\,d\mu$ for all $n\in\mathbb{N}$ and so

$$\limsup_{n\to\infty}\int_X f_n\,d\mu \le \int_X f\,d\mu.$$

Hence,

$$\int_X f\,d\mu \le \liminf_{n\to\infty}\int_X f_n\,d\mu \le \limsup_{n\to\infty}\int_X f_n\,d\mu \le \int_X f\,d\mu \implies \int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu.$$

**Problem 2.1.12**

a) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow [0, \infty]$ be a sequence of measurable positive functions. Let us suppose that $f_n(x) \searrow f(x)$ and that $\int_X f_k \, d\mu < \infty$ for some $k$. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

b) Let $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and $a > 0$. Let us define $f_n(x) = a_n/x$, for $x > a > 0$. Check that $f_n$ decreases uniformly to 0 but $\int f_n \, dm = \infty$ for all $n$.

*Hint:* a) Consider the sequence $g_n = f_k - f_{k+n}$.

*Solution:* a) Let $g_n = f_k - f_{k+n}$ for $n \in \mathbb{N}$. Then $\{g_n\}$ is increasing and $g_n \nearrow f_k - f := g$ as $n \to \infty$. By the monotone convergence theorem:

$$\int_X g \, d\mu = \lim_{n\to\infty} \int_X g_n d\mu \implies \int_X f_k \, d\mu - \int_X f \, d\mu = \int_X f_k \, d\mu - \lim_{n\to\infty} \int_X f_{k+n} \, d\mu \,.$$

As $\int_X f_k d\mu < \infty$, subtracting it from both members of the last equality we obtain that

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_{k+n} \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu \,.$$

b) We have that $f_1(x) \geq f_2(x) \geq \cdots \geq f_n(x) \geq \cdots \searrow 0$ as $n \to \infty$ for all $x \in (a, \infty)$. The convergence is even uniform:

$$0 < f_n(x) = \frac{a_n}{x} \leq \frac{a_n}{a} < \varepsilon \iff a_n < a\varepsilon$$

and, since $\lim_{n\to\infty} a_n = 0$, this happens for $n \geq n_0(\varepsilon)$, independently on $x \in (a, \infty)$. But, by the monotone convergence theorem,

$$\int_a^\infty f_n(x) \, dx = \lim_{N\to\infty} \int_a^N \frac{a_n}{x} \, dx = \lim_{N\to\infty} a_n [\log x]_{x=a}^{x=N} = \lim_{N\to\infty} a_n (\log N - \log a) = \infty$$

for all $n \in \mathbb{N}$. As $\int_a^\infty 0 \, dx = 0$, we conclude that part a) fails if the functions are not integrable even in the case of uniform convergence.

**Problem 2.1.13** Let $g : (X, \mathcal{A}, \mu) \longrightarrow [0, \infty]$ be an integrable function. Let $\{E_n\}$ be a decreasing sequence of sets such that $\cap_{n=1}^\infty E_n = \varnothing$. Prove that $\lim_{n\to\infty} \int_{E_n} g \, d\mu = 0$.

*Solution:* As $\{E_n\}_{n=1}^\infty$ is decreasing we have that $\{g \, \chi_{E_n}\}_{n=1}^\infty$ is also decreasing and $\{g \, \chi_{E_n}\}_{n=1}^\infty \searrow g \, \chi_{\cap_n E_n} = g \, \chi_\varnothing = 0$. But, as $\int_X g \, \chi_{E_1} \, d\mu \leq \int_X g \, d\mu < \infty$, we can apply part a) of problem 2.1.12, and so

$$\lim_{n\to\infty} \int_{E_n} g \, d\mu = \lim_{n\to\infty} \int_X g \, \chi_{E_n} \, d\mu = \int_X \lim_{n\to\infty} g \, \chi_{E_n} \, d\mu = \int_X 0 \, d\mu = 0 \,.$$

**Problem 2.1.14** Prove that for all $a > 0$, the function $f(x) = e^{-x} x^{a-1}$ is Lebesgue-integrable in $[0, \infty]$.

*Hints:* $e^{-x} \leq 1$ for $x \in [0, 1]$; $f$ is continuous in any bounded interval $[1, M]$; $\lim_{x\to\infty} x^{a-1} e^{-x/2} = 0$.

*Solution:* a) *Case* 1: $a \geq 1$. Then, for all $M > 0$, $f$ is continuous in $[0, M]$ and so $f \in L^1[0, M]$. On the other hand, applying L'Hopital rule $[a]$ times:

$$\lim_{x\to\infty} \frac{x^{a-1}}{e^{x/2}} = \lim_{x\to\infty} \frac{(a-1)\cdots(a-[a])x^{a-[a]-1}}{\frac{1}{2^{[a]}}e^{x/2}} = 0\,.$$

Therefore, there exists $M > 0$ such that $x^{a-1}e^{-x/2} < 1$ for all $x \in [M, \infty)$, and so, by part b.1) of problem 2.1.8,

$$\int_M^\infty e^{-x}x^{a-1}dx \leq \int_M^\infty e^{-x}e^{x/2}dx = \int_M^\infty e^{-x/2}dx \leq \int_0^\infty e^{-x/2}dx = 2 < \infty\,.$$

Hence, $f \in L^1(0, \infty)$.
b) *Case* 1: $0 < a < 1$. Then $0 < 1 - a < 1$ and, as $e^{-x} \leq e^0 = 1$ for all $x > 0$, we have that

$$\int_0^1 e^{-x}x^{a-1}dx \leq \int_0^1 x^{a-1}dx = \int_0^1 \frac{1}{x^{1-a}}\,dx < \infty$$

by part b.3) of problem 2.1.8, since $0 < 1 - a < 1$. Hence, $f \in L^1(0, 1)$. Also, since $a - 1 < 0$,

$$\lim_{x\to\infty} x^{a-1}e^{-x/2} = 0$$

and we conclude like in part a) that $f \in L^1[M, \infty)$ for some $M > 0$. Finally, as $f$ is continuous in $[1, M]$ we have that $f$ is bounded there and so $f \in L^1[1, M]$. Hence, $f \in L^1(0, \infty)$ also in this case.

**Problem 2.1.15** Let $f_n : [0, 1] \longrightarrow [0, \infty)$ be a sequence of positive functions defined by

$$f_n(x) = \begin{cases} n\,, & \text{if } 0 \leq x \leq 1/n\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Check that $f_n \to 0$ pointwise when $x > 0$ but $\int f_n\,dm = 1$. Interpret why this may happen.

*Solution:* Given $x \in (0, 1]$, we have that $x > 1/n$ for all $n \geq n_0(x)$. Hence, $f_n(x) = 0$ for all $n \geq n_0(x)$ and so $\lim_{n\to\infty} f_n(x) = 0$. However,

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n\,dx = n \cdot \frac{1}{n} = 1\,, \qquad \text{for all } n \in \mathbb{N}\,.$$

Hence,

$$1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \big(\lim_{n\to\infty} f_n(x)\big)\,dx = 0\,,$$

in spite of $\exists \lim_{n\to\infty} f_n$. This fact shows that the monotonicity in the the monotone convergence theorem is necessary.

**Problem 2.1.16** Let $\mathcal{M}$ be the $\sigma$-algebra of Lebesgue-measurable sets in $[0, \infty)$. We define in $\mathcal{M}$ the measure $\mu$ as

$$\mu(E) = \int_E \frac{1}{1+x}\,dx\,.$$

Check that $\mu$ is a Borel-Stieltjes measure and calculate the corresponding distribution function $F$. Find a function $f(x)$ such that $\int f\,d\mu < \infty$ but $\int f\,dm = \infty$, being $m$ the Lebesgue measure.

*Hint:* $F(t) = \log(1+t)\,\chi_{[0,\infty]}(t)$; $f(x) = 1/(1+x)$.
*Solution:* $\mu$ is a Radon measure in $[0,\infty)$, since for all $M > 0$,

$$\mu([0,M)) = \int_0^M \frac{1}{1+x}\,dx = [\log(1+x)]_{x=0}^{x=M} = \log M < \infty\,.$$

Hence, by problem 1.3.12, $\mu$ is a Borel-Stieltjes measure. Since

$$\mu([a,b)) = \int_a^b \frac{1}{1+x}\,dx = [\log(1+x)]_{x=a}^{x=b} = \log(1+b) - \log(1+a)$$

we see that $\mu = \mu_F$ with $F$ the distribution function $F(x) = \log(1+x)$.
Now, let $f(x) = 1/(1+x)$. Then, by the monotone convergence theorem,

$$\int_{[0,\infty)} f\,dm = \int_0^\infty \frac{1}{1+x}\,dx = \lim_{N\to\infty}\int_0^N \frac{1}{1+x}\,dx = \lim_{N\to\infty}[\log(1+x)]_{x=0}^{x=N} = \lim_{N\to\infty}(\log(1+N) = \infty\,.$$

But, $\mu$ is a measure given by the density function $f$, using problem 2.1.3 and the monotone convergence theorem, we get

$$\begin{aligned}\int_{[0,\infty)} f\,d\mu &= \int_0^\infty \frac{1}{1+x}\,\frac{1}{1+x}\,dx = \int_0^\infty \frac{1}{(1+x)^2}\,dx = \lim_{N\to\infty}\int_0^N \frac{1}{(1+x)^2}\,dx\\ &= \lim_{N\to\infty}\Big[-\frac{1}{1+x}\Big]_{x=0}^{x=N} = 1 - \lim_{N\to\infty}\frac{1}{1+N} = 1 - 0 = 1 < \infty\,.\end{aligned}$$

**Problem 2.1.17** Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be measure spaces and let $A, A_i \in \mathcal{A}$, $B, B_i \in \mathcal{B}$ $(i \in \mathbb{N})$ be sets such that

$$A \times B = \bigcup_{i=1}^\infty (A_i \times B_i)\,, \qquad A_i \times B_i \text{ disjoint sets.}$$

Prove that

$$\mu(A)\nu(B) = \sum_{i=1}^\infty \mu(A_i)\nu(B_i)\,.$$

*Hint:* Use that for a positive sequence of functions: $\sum_n \int f_n = \int \sum_n f_n$.
*Solution:* Let $\mu\otimes\nu$ be the product measure of $\mu$ and $\nu$. As the sets $A_i \times B_i$ are pairwise disjoint we have that

$$\chi_{A\times B}(x,y) = \sum_{i=1}^\infty \chi_{A_i\times B_i}(x,y)$$

and so,

$$\begin{aligned}\mu(A)\,\nu(B) &= \int_{X\times Y} \chi_{A\times B}\,d(\mu\otimes\nu) = \int_{X\times Y}\sum_{i=1}^\infty \chi_{A_i\times B_i}\,d(\mu\otimes\nu)\\ &= \sum_{i=1}^\infty \int_{X\times Y}\chi_{A_i\times B_i}\,d(\mu\otimes\nu) = \sum_{i=1}^\infty (\mu\otimes\nu)(A_i\times B_i) = \sum_{i=1}^\infty \mu(A_i)\,\nu(B_i)\,.\end{aligned}$$

**Problem 2.1.18** Prove Borel-Cantelli Lemma (see Problem 1.2.11) using the the monotone convergence theorem.

*Hint:* Consider the function $\sum_{n=1}^{\infty} \chi_{A_n}$.
*Solution:* Let $F = \sum_{n=1}^{\infty} \chi_{A_n}$. Then

$$\int_X F\,d\mu = \sum_{n=1}^{\infty} \int_X \chi_{A_n}\,d\mu = \sum_{n=1}^{\infty} \mu(A_n) < \infty$$

by hypothesis. Therefore we must have that $\mu(\{x \in X : F(x) = \infty\}) = 0$, since on the contrary we would have $\int_X F\,d\mu = \infty$. But $F(x) = \infty \iff x \in A_n$ for infinitely many $n$, and so

$$\mu(\{x \in X : x \in A_n \text{ for infinitely many } n\}) = 0\,.$$

**Problem 2.1.19** Let $A = [0,1] \cap \mathbb{Q}$. Then we can write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let us define the functions $f_n : [0,1] \longrightarrow \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1\,, & \text{if } x \in \{a_1, \ldots, a_n\}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Prove that $f_n$ is Riemann-integrable and calculate $f(x) = \lim_{n\to\infty} f_n(x)$. Are $f_n$ and $f$ Lebesgue-integrable functions?

*Solution:* We have that $f_n$ is only discontinuous at the points $a_1, a_2, \ldots, a_n$. Hence, $f_n$ is bounded and continuous almost everywhere with respect to Lebesgue measure in $[0,1]$ and therefore $f_n$ is Riemann-integrable and so, Lebesgue-integrable. Besides $\lim_{n\to\infty} f_n(x) = \chi_A(x) = f(x)$. Since $f_n = f = 0$ almost everywhere for each $n$ since $m(A) = 0$ because $A$ is countable, the integrals of $f_n$ and $f$ are all zero.

**Problem 2.1.20** With the notation of the problem above, let $F(x)$ be the function

$$F(x) = \begin{cases} \frac{1}{k}\,, & \text{if } x = a_k\,, \\ 0\,, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

Show that the function $F$ is Riemann-integrable on any bounded interval $[a,b]$ and find $\int_a^b F(x)\,dx$.

*Solution:* $F$ is bounded and continuous in $\mathbb{R} \setminus \mathbb{Q}$ and so, $F$ is almost everywhere continuous since $m(\mathbb{Q}) = 0$ because $\mathbb{Q}$ is countable. Hence, $F$ is Riemann-integrable on $[a,b]$ and $\int_a^b F(x)\,dx = 0$ since $F(x) = 0$ almost everywhere with respect to Lebesgue measure.