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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.2: Integration of general functions

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2 Integration Theory

2.2. Integration of general functions

Problem 2.2.1 Let $f_n : [0,1] \longrightarrow [-1,1]$ be a sequence of continuous functions such that $f_n(x) \to 0$ almost everywhere with respect to Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0\,.$$

Hint: The functions f_n are uniformly bounded.

Solution: As $|f_n(x)| \le 1$ for all $n \in \mathbb{N}$ and $1 \in L^1[0,1]$ we can apply the dominated convergence theorem and so

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 (\lim_{n\to\infty} f_n(x))\, dx = \int_0^1 0\, dx = 0\,.$$

Problem 2.2.2 Let $(X, \mathcal{A}, \mu)$ be a finite space measure: $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n(x) \to f(x)$ uniformly in X. Prove that $f \in L^1(\mu)$ and that

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu\,.$$

Hint: Uniform convergence implies that the sequence f_n is uniformly-Cauchy.

Solution: As f_n tends uniformly to f, we have that there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < 1\,, \qquad \forall n \ge n_0,\ \forall x \in X\,.$$

Hence, by the triangle inequality,

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < 1 + |f_n(x)|\,, \qquad \forall n \ge n_0,\ \forall x \in X\,.$$

and so

$$\int_X |f|\, d\mu \le \int_X (1 + |f_{n_0}|)\, d\mu = \mu(X) + \int_X |f_{n_0}|\, d\mu < \infty \implies f \in L^1(\mu)\,.$$

Also, again by the triangle inequality,

$$|f_n(x)| \le |f(x) - f_n(x)| + |f(x)| \le 1 + |f(x)| \in L^1(\mu)\,, \qquad \forall n \ge n_0,\ \forall x \in X\,,$$

and using the dominated convergence theorem we get that $\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu$.

Problem 2.2.3* Let $f_n : (\mathbb{R}, \mathcal{M}, m) \longrightarrow [0,\infty)$ be a sequence of positive Lebesgue-measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in \mathbb{R}$ and, besides, $\int_{\mathbb{R}} f_n\, dx = \int_{\mathbb{R}} f\, dx = 1$ for all $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^x f_n\, dx = \int_{-\infty}^x f\, dx\,, \qquad \text{for all } x \in \mathbb{R}\,.$$

Hint: Consider the functions $\min(f_n, f)$ and use an adequate convergence theorem. Recall that $\min(x,y) = \frac{x+y-|x-y|}{2}$.

Solution: Let $F_n(x) = \int_{-\infty}^x f_n\,dx$ and $F(x) = \int_{-\infty}^x f\,dx$. We have that, for all $x \in \mathbb{R}$,

$$|F_n(x) - F(x)| = \Big|\int_{-\infty}^x (f_n - f)\,dx\Big| \le \int_{-\infty}^x |f_n - f|\,dx \le \int_{\mathbb{R}} |f_n - f|\,dx\,. \tag{1}$$

On the other hand we have that, as $n \to \infty$,

$$f_n \longrightarrow f \text{ a.e.} \implies \min(f_n, f) \longrightarrow f \text{ a.e.}.$$

Also $\min(f_n, f) \le f \in L^1(\mathbb{R})$. Hence by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}} \min(f_n, f)\,dx = \int_{\mathbb{R}} f\,dx\,,$$

and, as $|f_n - f| = f_n + f - 2\min(f_n, f)$, we obtain that

$$\int_{\mathbb{R}} |f_n - f|\,dx = \int_{\mathbb{R}} f_n\,dx + \int_{\mathbb{R}} f\,dx - 2\int_{\mathbb{R}} \min(f_n, f)\,dx \to \int_{\mathbb{R}} f\,dx + \int_{\mathbb{R}} f\,dx - 2\int_{\mathbb{R}} f\,dx = 0$$

as $n \to \infty$. Hence, using (1), we get that $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in \mathbb{R}$.

Problem 2.2.4 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{R}$ be an integrable function.

a) Prove Markov's inequality:

$$\mu(\{x \in X : \ |f(x)| \ge \varepsilon\}) \le \frac{1}{\varepsilon}\int_X |f|\,d\mu\,.$$

b) Using Markov's inequality, show that if f is a measurable function, then

$$\begin{array}{llcl} \text{b1)} & \int |f|\,d\mu \ = 0 & \iff & \mu(f \ne 0) = 0\,, \\ \text{b2)} & \int |f|\,d\mu \ < \infty & \implies & \mu(|f| = \infty) = 0\,. \end{array}$$

Give an example showing that it is possible to have that

$$\int |f|\,d\mu = \infty \qquad \text{and} \qquad \mu(|f| = \infty) = 0\,.$$

Hints: a) $1 \le \frac{1}{\varepsilon}|f|$ on the set $\{x \in X : |f(x)| \ge \varepsilon\}$. b1) If $\int |f|\,d\mu = 0$, then $\mu(|f(x)| \ge 1/n)\}) = 0$ for all $n \in \mathbb{N}$. b2) If $\int |f|\,d\mu < \infty$, then $\{|f| = \infty\} \subset \{|f| \ge n\}$ for all $n \in \mathbb{N}$.

Solution: a) $\mu(\{x \in X : \ |f(x)| \ge \varepsilon\}) = \displaystyle\int_{\{|f| \ge \varepsilon\}} 1\,d\mu \le \int_{\{|f| \ge \varepsilon\}} \frac{1}{\varepsilon}|f|\,d\mu \le \frac{1}{\varepsilon}\int_X |f|\,d\mu.$

b1) $(\Leftarrow)$ $\displaystyle\int_X |f|\,d\mu = \int_{\{|f|=0\}} |f|\,d\mu + \int_{\{|f|\ne 0\}} |f|\,d\mu = 0 + 0 = 0.$

$(\Rightarrow)$ Using part a) we have that $\mu(\{x \in X : |f| \ge 1/n\}) = 0$ for all $n \in \mathbb{N}$, and so

$$\mu(\{x \in X : f(x) \ne 0\}) = \mu\Big(\bigcup_{n=1}^\infty \{x \in X : |f(x)| \ge 1/n\}\Big) \le \sum_{n=1}^\infty \mu(\{x \in X : |f| \ge 1/n\}) = 0\,.$$

b2) Using part a) we have that for all $n \in \mathbb{N}$, and since $\int_X |f|\,d\mu < \infty$:

$$\mu(\{x \in X : f(x) = \infty\}) \le \mu(\{x \in X : |f| \ge n\}) \le \frac{1}{n}\int_X |f|\,d\mu \to 0\,, \qquad \text{as } n \to \infty\,.$$

Hence,$\mu(\{x \in X : f(x) = \infty\}) = 0$.

The converse is false: Take $X = [1, \infty)$ and $f(x) = 1/x$. Then $\{x : |f(x)| = \infty\} = \varnothing$ and so, $\mu(\{x : |f(x)| = \infty\}) = 0$ but $\int_X |f|\,dx = \infty$.

Problem 2.2.5 Prove that the function $f(x) = \dfrac{\sin x}{x}$ is not Lebesgue-integrable in $(0, \infty)$.
Hint: Divide $(0, \infty)$ in the intervals $(n\pi, (n+1)\pi]$ $(n \geq 0)$.
Solution: We have that

$$\int_0^\infty \Big|\frac{\sin x}{x}\Big|\, dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\, dx \geq \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin x|\, dx\,.$$

But the function $|\sin x|$ is π-periodic and so

$$\int_0^\infty \Big|\frac{\sin x}{x}\Big|\, dx \geq \frac{1}{\pi}\Big(\sum_{n=0}^{\infty} \frac{1}{n+1}\Big) \int_0^\pi \sin x\, dx = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} = \infty$$

and so, $\int_0^\infty |\sin x/x|\, dx = \infty$ and $f \notin L^1(0, \infty)$.

Problem 2.2.6 Discuss whether the following functions are Lebesgue integrable or not. Give an argument of why they are not, or find the value of the integral:

a) $f(x) = \dfrac{1 - \cos x}{x(1 + x^2)}$ for $x \in (0, \infty)$.

b) $g(x) = \sin x + \cos x$ for $x \in \mathbb{R}$.

Hints: a) On $(0, \delta)$ we have $|f(x)| \leq Cx/(1+x^2) \in L^1(0, \delta)$ and on (δ, ∞) we have $|f(x)| \leq 2/x^3 \in L^1(\delta, \infty)$. b) $|\sin x + \cos x|$ is π-periodic, $f(x) > 0$ on $(-\pi/4, 3\pi/4)$ and $\int_{-\pi/4}^{3\pi/4} |\sin x + \cos x|\, dx = 2\sqrt{2} > 0$.
Solution: a) As $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2/2} = 1$ we have that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Big|\frac{1 - \cos x}{x^2/2}\Big| < 1 + \varepsilon\,, \qquad \text{if } |x| < \delta\,,$$

Therefore, since $x/(1 + x^2)$ is bounded in $[0, \delta]$ (because it is continuous there), if $x \in (0, \delta)$, then

$$|f(x)| < \frac{(1 + \varepsilon)x^2/2}{x(1 + x^2)} = \frac{1 + \varepsilon}{2}\,\frac{x}{1 + x^2} \in L^1(0, \delta)\,.$$

On the other hand, if $x \in (\delta, \infty)$ then, by part b2) of problem 2.1.8:

$$|f(x)| \leq \frac{2}{x(1 + x^2)} < \frac{2}{x^3} \in L^1(\delta, \infty)\,.$$

Hence, $f \in L^1(0, \infty)$.
b) If $x \in [-\pi, \pi]$, then $g(x) = 0 \iff \tan x = -1 \iff x = -\pi/4$ or $x = 3\pi/4$. Hence,$g(x) \geq 0$ in $[-\pi/4, 3\pi/4]$ and, as g is π-periodic and

$$\int_{-\pi/4}^{3\pi/4} g(x)\, dx = \int_{-\pi/4}^{3\pi/4} (\sin x + \cos x)\, dx = [-\cos x + \sin x]_{x=-\pi/4}^{x=3\pi/4} = 2\sqrt{2}\,,$$

we conclude that $g \notin L^1(\mathbb{R})$, since

$$\int_{\mathbb{R}} |g(x)|\, dx = \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{4}+n\pi}^{\frac{3pi}{4}+n\pi} |\sin x + \cos x|\, dx = \sum_{n\in\mathbb{Z}} 2\sqrt{2} = \infty\,.$$

Problem 2.2.7 It is easy to guess the limits

a) $\displaystyle\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{x/2}dx\,,$

b) $\displaystyle\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx\,.$

Prove that your guesses are correct.

Solution: a) Let $f_n(x)=(1-\frac{x}{2})^n e^{x/2}\chi_{[0,n]}(x)$. As $\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n=e^{-x}$, we have $\lim_{n\to\infty} f_n(x)=e^{-x/2}$. Hence, in view of problem 2.1.8 part b1), we guess that:

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{x/2}dx=\lim_{n\to\infty}\int_0^\infty f_n(x)\,dx=\int_0^\infty\left(\lim_{n\to\infty} f_n(x)\right)dx=\int_0^\infty e^{-x/2}dx=2.$$

To prove it, we will show that $|f_n(x)|\le e^{-x/2}\in L^1(0,\infty)$ and then our conjecture will be a consequence of the dominated convergence theorem. To do that it is enough to prove that $(1-\frac{x}{n})^n\le e^{-x}$ if $x\in[0,n]$. This inequality is equivalent to $n\log(1-\frac{x}{n})\le -x$. If we define $F(x):=x+n\log(1-\frac{x}{n})$ for $x\in[0,n]$, then we must prove that $F(x)\le 0$ for $x\in[0,n]$. But

$$F'(x)=1-\frac{1}{1-\frac{x}{n}}=-\frac{x/n}{1-\frac{x}{n}}\le 0 \implies F \text{ is decreasing} \implies F(x)\le F(0)=0\,.$$

b) Let $g_n(x)=(1+\frac{x}{2})^n e^{-2x}\chi_{[0,n]}(x)$. As $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^{x}$, we have $\lim\limits_{n\to\infty} g_n(x)=e^{-x}$. Hence, in view of problem 2.1.8 part b1), we guess that:

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx=\lim_{n\to\infty}\int_0^\infty g_n(x)\,dx=\int_0^\infty\left(\lim_{n\to\infty} g_n(x)\right)dx=\int_0^\infty e^{-x}\,dx=1.$$

To prove it, we will show that $|g_n(x)|\le e^{-x}\in L^1(0,\infty)$ and then our conjecture will be a consequence of the dominated convergence theorem. To do that it is enough to prove that $(1+\frac{x}{n})^n\le e^{x}$ if $x\in[0,n]$. This inequality is equivalent to $n\log(1+\frac{x}{n})\le x$. If we define $G(x):=x-n\log(1+\frac{x}{n})$ for $x\in[0,n]$, then we must prove that $G(x)\ge 0$ for $x\in[0,n]$. But

$$G'(x)=1-\frac{1}{1+\frac{x}{n}}=\frac{x/n}{1+\frac{x}{n}}\ge 0 \implies G \text{ is increasing} \implies G(x)\ge G(0)=0\,.$$

Problem 2.2.8 Let $(X,\mathcal{A},\mu)$ be a measure space and let $f_n : X\longrightarrow\mathbb{R}$ be a sequence of measurable functions such that

$$\sum_{n=1}^\infty\int_X |f_n|\,d\mu<\infty\,.$$

Prove that:

a) The series $\sum_n f_n$ converges almost everywhere in X to a function $f: X\longrightarrow\mathbb{R}$:

$$\sum_{n=1}^\infty f_n(x)=f(x)\,,\qquad \text{for almost every } x\in X\,.$$

b) $f\in L^1(\mu)$.

c) $\displaystyle\int_X f\,d\mu = \sum_{n=1}^{\infty}\int_X f_n\,d\mu\,.$

Hints: a) Consider the function $F(x) := \sum_{n=1}^{\infty}|f_n(x)| \in L^1(X)$, why? Then $|f(x)| \leq F(x) < \infty$ almost everywhere (use problem 2.2.4). b) It follows easily from a). c) $g_n := f_1 + \cdots + f_n$ verifies $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. and $|g_n| \leq F$. Use a convergence theorem.

Solution: a) Let $F(x) = \sum_{n=1}^{\infty}|f_n(x)| \in [0,\infty]$. Then, as a consequence of monotone convergence theorem

$$\begin{aligned}\int_X F\,d\mu &= \int_X \lim_{N\to\infty}\sum_{n=1}^{N}|f_n(x)|\,d\mu(x) = \lim_{N\to\infty}\int_X\sum_{n=1}^{N}|f_n(x)|\,d\mu(x)\\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_X|f_n(x)|\,d\mu(x) = \sum_{n=1}^{\infty}\int_X|f_n|\,d\mu < \infty\,,\end{aligned}$$

by hypothesis. Therefore:

$$F\in L^1(\mu) \implies F(x)<\infty \text{ a.e.} \implies \sum_{n=1}^{\infty}|f_n(x)|<\infty \text{ a.e.} \implies \sum_{n=1}^{\infty}f_n(x) \text{ converges a.e.}$$

b) As $|f(x)| = \left|\sum_{n=1}^{\infty}f_n(x)\right| \leq \sum_{n=1}^{\infty}|f_n(x)| = F(x)\in L^1(\mu)$ we conclude that also $f\in L^1(\mu)$.

c) Let $s_N(x) = \sum_{n=1}^{N}f_n(x)$. Then:

$$|s_N(x)| \leq \sum_{n=1}^{N}|f_n(x)| \leq F(x)\in L^1(\mu) \qquad\text{and}\qquad s_N(x)\to f(x) \quad\text{as } N\to\infty\,,$$

and so, by the dominated convergence theorem:

$$\int_X f\,d\mu = \int_X\lim_{N\to\infty}s_N(x)\,d\mu(x) = \lim_{N\to\infty}\int_X s_N(x)\,d\mu(x) = \lim_{N\to\infty}\sum_{n=1}^{N}\int_X f_n\,d\mu = \sum_{n=1}^{\infty}\int_X|f_n|\,d\mu\,.$$

Problem 2.2.9 Prove that

$$\lim_{n\to\infty}\int_0^{\infty}\frac{dx}{(1+x/n)^n x^{1/n}} = 1\,.$$

Hint: $f_n(x) \leq \frac{1}{\sqrt{x}}\chi_{(0,1]}(x) + (1+x/2)^{-2}\chi_{(1,\infty)}(x) \in L^1(0,\infty)$ for $n\geq 2$ and so we can use dominated convergence.

Solution: Let $f_n(x) = \frac{1}{(1+x/n)^n x^{1/n}}$ for $x\in(0,\infty)$. Then, using problem 2.1.8, we have for $n\geq 2$

- If $x\in(0,1]$, then $f_n(x) \leq \frac{1}{x^{1/n}} \leq \frac{1}{x^{1/2}} \in L^1(0,1]$.
- If $x\in(1,\infty)$, then $f_n(x) \leq \frac{1}{(1+x/n)^n} \leq \frac{1}{(1+x/2)^2} \leq \frac{4}{x^2} \in L^1(1,\infty)$.

Hence,$f_n(x) \leq \frac{1}{\sqrt{x}}\chi_{(0,1]}(x) + \frac{4}{x^2}\chi_{(1,\infty)}(x) \in L^1(0,\infty)$ and, by the dominated convergence theorem and problem 2.2.8,

$$\lim_{n\to\infty}\int_0^{\infty}\frac{dx}{(1+x/n)^n x^{1/n}} = \lim_{n\to\infty}\int_0^{\infty}f_n(x)\,dx = \int_0^{\infty}\Big(\lim_{n\to\infty}f_n(x)\Big)\,dx = \int_0^{\infty}\frac{1}{e^x}\,dx = 1\,.$$

Problem 2.2.10 Let us consider the functions

$$f_n(x) = \frac{nx-1}{(x\log n+1)(1+nx^2\log n)}\,, \qquad x\in(0,1]\,.$$

Prove that

$$\lim_{n\to\infty} f_n(x) = 0\,, \qquad \text{but} \quad \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2}\,.$$

What is the relevance of this result?

Hint: Prove that $\dfrac{nx-1}{(x\log n+1)(1+nx^2\log n)} = \dfrac{-1}{x\log n+1} + \dfrac{nx}{(n\log n)x^2+1}\,.$

Solution: First of all, we have that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\frac{x}{\log^2 n} - \frac{1}{n\log^2 n}}{\left(x+\frac{1}{\log n}\right)\left(x^2+\frac{1}{n\log n}\right)} = \frac{0}{x\cdot x^2} = 0\,.$$

Now, we decompose $f_n(x)$ into simple fractions:

$$f_n(x) = \frac{A_n}{x\log n+1} + \frac{B_n x + C_n}{1+nx^2\log n}\,.$$

Eliminating denominators we obtain the equivalent equation

$$nx-1 = A_n(1+nx^2\log n) + (B_n x + C_n)(x\log n+1)$$

and from this, it is easy to obtain: $A_n=-1$, $B_n=n$ and $C_n=0$. Hence,

$$\begin{aligned}\int_0^1 f_n(x)\,dx &= \int_0^1 \frac{-1}{x\log n+1}\,dx + \int_0^1 \frac{nx}{1+nx^2\log n}\,dx \\ &= \Big[-\frac{\log(x\log n+1)}{\log n}\Big]_{x=0}^{x=1} + \Big[\frac{\log(1+nx^2\log n)}{2\log n}\Big]_{x=0}^{x=1} \\ &= -\frac{\log(\log n+1)}{\log n} + \frac{\log(1+n\log n)}{2\log n}\,,\end{aligned}$$

and so, using L'Hopital rule, we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_n(x)\,dx &= -\lim_{t\to\infty}\frac{\log(\log t+1)}{\log t} + \lim_{t\to\infty}\frac{\log(1+t\log t)}{2\log t} = \lim_{t\to\infty}\frac{\frac{-1/t}{\log t+1}}{1/t} + \lim_{t\to\infty}\frac{\frac{1+\log t}{1+t\log t}}{2/t} \\ &= \lim_{t\to\infty}\frac{-1}{\log t+1} + \frac{1}{2}\lim_{t\to\infty}\frac{t(1+\log t)}{1+t\log t} = 0 + \frac{1}{2}\lim_{t\to\infty}\frac{\frac{1}{\log t}+1}{\frac{1}{t\log t}+1} = \frac{1}{2}\cdot 1 = \frac{1}{2}\,.\end{aligned}$$

Hence,we have obtained that $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1(\lim_{n\to\infty} f_n(x))\,dx$ and as a consequence we obtain that $f_n(x)\to 0$ as $n\to\infty$ but not monotonically and also that we can not dominate the functions f_n by an integrable function in $(0,1]$.

Problem 2.2.11 Consider $a>0$.

a) Prove that for each $x\geq a$ the function $v(t) := \dfrac{t}{1+t^2x^2}$ decreases for $t\geq 1/a$.

b) Find an upper bound of the function

$$f_n(x) = \frac{n}{1+n^2x^2}\,, \qquad x \geq a\,,\ n \geq 1/a\,,$$

by a function which just depends on x and a.

c) Calculate

$$L = \lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\,dx\,,$$

and say what theorem you used.

d) Calculate L using monotone convergence theorem and Barrow's rule in the cases $a > 0$, $a = 0$, $a < 0$.

Solution: a) We have that

$$v'(t) = \frac{1-t^2x^2}{(1+t^2x^2)^2} = 0 \qquad \Leftrightarrow \qquad t^2 = \frac{1}{x^2}$$

and therefore, since $x \geq a > 0$,

$$t \geq \frac{1}{a} \implies t \geq \frac{1}{x} \implies t^2 \geq \frac{1}{x^2} \implies 1 - x^2t^2 \leq 0 \implies v'(t) \leq 0\,.$$

Hence,$v(t)$ decreases in the interval $[1/a, \infty)$.

b) As a consequence of a)

$$v(t) \leq v(1/a) = \frac{a}{a^2+x^2} \qquad \text{if } t \geq 1/a\,.$$

Therefore, if $n \geq 1/a$,

$$f_n(x) = \frac{n}{1+n^2x^2} = v(n) \leq v(1/a) = \frac{a}{a^2+x^2}\,.$$

c) As $F(x) = \dfrac{a}{a^2+x^2} \in L^1(a,\infty)$, by the dominated convergence theorem:

$$\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\,dx = \int_a^\infty \lim_{n\to\infty} \frac{n}{1+n^2x^2}\,dx = \int_a^\infty 0\,dx = 0\,.$$

d) Using the monotone convergence theorem and Barrow's rule, we have:

$$\begin{aligned}\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\,dx &= \lim_{n\to\infty} \int_a^\infty \Big(\lim_{N\to\infty} \frac{n}{1+n^2x^2}\,\chi_{[a,N]}\Big)dx = \lim_{n\to\infty}\lim_{N\to\infty} \int_a^N \frac{n}{1+n^2x^2}\,dx \\ &= \lim_{n\to\infty}\lim_{N\to\infty} [\arctan(nx)]_{x=a}^{x=N} = \lim_{n\to\infty}\lim_{N\to\infty} \big(\arctan(nN) - \arctan(an)\big) \\ &= \lim_{n\to\infty} \Big(\frac{\pi}{2} - \arctan(an)\Big).\end{aligned}$$

Hence, $L = \frac{\pi}{2} - \frac{\pi}{2} = 0$ if $a > 0$, $L = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ if $a = 0$ and $L = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$ if $a < 0$.

Problem 2.2.12 Calculate $L = \displaystyle\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\,dx\,.$

Hint: $\{f_n\}$ is a decreasing sequence and $f_2 \in L^1((0,\infty))$. So we can use a convergence theorem.
Solution: a) We have that

$$f_n(x) \geq f_{n+1}(x) \quad \Leftrightarrow \quad (1+nx^2)(1+x^2) \geq 1+(n+1)x^2 \quad \Leftrightarrow \quad nx^4 \geq 0$$

and this is obviously true.

b) First, observe that

$$\begin{aligned}\int_0^\infty f_2(x)\,dx &= \int_0^\infty \frac{1+2x^2}{(1+x^2)^2}\,dx \leq \int_0^1 (1+2x^2)\,dx + \int_1^\infty \frac{1+2x^2}{x^4}\,dx \\ &= \int_0^1 (1+2x^2)\,dx + \int_1^\infty \frac{1}{x^4}\,dx + 2\int_1^\infty \frac{1}{x^2}\,dx < \infty\,.\end{aligned}$$

Hence, using the monotone convergence theorem for decreasing sequences or the dominated convergence theorem:

$$L = \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty \lim_{n\to\infty} f_n(x)\,dx = \int_0^\infty 0\,dx = 0,$$

since, for $x \neq 0$,

$$0 \leq \lim_{n\to\infty} f_n(x) \leq \lim_{n\to\infty} \frac{1+nx^2}{1+nx^2+\frac{n(n-1)}{2}x^4} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+\frac{x^2}{n}}{\frac{1}{n^2}+\frac{x^2}{n}+\frac{n-1}{2n}x^4} = \frac{0+0}{0+0+\frac{x^4}{2}} = 0\,.$$

Problem 2.2.13 Prove that $\lim_{n\to\infty} \int_0^1 \frac{\log(n+x)}{n}\, e^{-x}\cos x\,dx = 0$.

Solution: Let $f_n(x) = \frac{\log(n+x)}{n}\, e^{-x}\cos x$ for $x \in [0,1]$. First of all, using Stolz criterion we have, for all $x \in [0,1]$, that

$$\lim_{n\to\infty} \frac{\log(x+n)}{n} = \lim_{n\to\infty} \frac{\log(x+n+1)-\log(x+n)}{(n+1)-n} = \lim_{n\to\infty} \log\frac{x+n+1}{x+n} = \log 1 = 0\,.$$

Also

$$|f_n(x)| \leq \frac{\log(n+1)}{n} \leq 1 \in L^1([0,1])$$

and so we can apply the dominated convergence theorem:

$$\lim_{n\to\infty} \int_0^1 \frac{\log(n+x)}{n}\, e^{-x}\cos x\,dx = \int_0^1 \Big(\lim_{n\to\infty} \frac{\log(n+x)}{n}\Big) e^{-x}\cos x\,dx = \int_0^1 0\,dx = 0\,.$$

Problem 2.2.14 Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be the sequence of measurable functions defined by

$$f_n(x) = \begin{cases} n\cos nx\,, & \text{if } x \in [-\frac{\pi}{2n}, \frac{\pi}{2n}]\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Study whether

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f_n(x)\,dx = \int_{-\pi}^{\pi} \lim_{n\to\infty} f_n(x)\,dx$$

or not. Can be applied in this case the monotone convergence theorem or the Lebesgue dominated convergence theorem?

Solution: If $x \neq 0$, then eventually $x \notin [-\frac{\pi}{2n}, \frac{\pi}{2n}]$ for n large enough and so $f_n(x) = 0$ for $n \geq n_0(x)$. Hence, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \neq 0$. On the other hand,

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{-\frac{\pi}{2n}}^{\frac{\pi}{2n}} n\cos nx\,dx = [\sin nx]_{x=-\frac{\pi}{2n}}^{x=\frac{\pi}{2n}} = \sin\frac{\pi}{2} - \Big(-\sin\frac{\pi}{2}\Big) = 1 - (-1) = 2\,.$$

Therefore,

$$2 = \lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx \neq \int_{\mathbb{R}}\Big(\lim_{n\to\infty} f_n(x)\Big)\,dx = 0$$

and as a consequence we obtain that $f_n(x) \to 0$ as $n \to \infty$ but not monotonically and also that we can not dominate the functions f_n by an integrable function in $\mathbb{R}$.

Problem 2.2.15 Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ the measure space defined by

$$\mu(A) = \operatorname{card}(A \cap \mathbb{N})\,, \qquad A \in \mathcal{B}(\mathbb{R})\,.$$

Prove that $f(x) = x\sin(\pi x)$ is μ-integrable but not Lebesgue-integrable.

Solution: We have that μ is the counting measure on $\mathbb{N}$ and

$$\int_{\mathbb{R}} |f(x)|\,d\mu(x) = \sum_{n=0}^{\infty} |n\sin(n\pi)| = 0 \implies f \in L^1(\mu)\,.$$

On the other hand, since $f(x)$ is even

$$\int_{\mathbb{R}} |f(x)|\,d\mu(x) = 2\int_0^{\infty} |f(x)|\,dx = 2\sum_{n=0}^{\infty}\int_n^{n+1} |f(x)|\,dx\,.$$

Now, if $x \in (n, n+1)$, then $|f(x)| = x\,|\sin(\pi x)| = (-1)^n x\sin(\pi x)$ and integrating by parts, we have

$$\begin{aligned}\int_n^{n+1} |x\sin(\pi x)|\,dx &= -\frac{(-1)^n}{\pi}\Big[x\cos(\pi x)\Big]_{x=n}^{x=n+1} + \frac{(-1)^n}{\pi}\int_n^{n+1}\cos(\pi x)\,dx \\ &= -(-1)^n\Big[\frac{1}{\pi}\,x\cos(\pi x) - \frac{1}{\pi^2}\,\sin(\pi x)\Big]_{x=n}^{x=n+1} \\ &= -(-1)^n\Big(\frac{n+1}{\pi}\,(-1)^{n+1} - \frac{n}{\pi^2}\,(-1)^n\Big) = \frac{n+1}{\pi} + \frac{n}{\pi^2}\,.\end{aligned}$$

Hence,

$$\int_{\mathbb{R}} |f(x)|\,d\mu(x) = \frac{1}{\pi^2}\sum_{n=0}^{\infty}\big(\pi(n+1) + n\big) = \infty \qquad \implies \qquad f \notin L^1(m)\,.$$

Problem 2.2.16

a) Prove that the sequence of functions

$$f_n(t) = \Big(1 + \frac{t}{n}\Big)^n, \qquad t \geq 0\,,$$

verify that $f_3(t) \leq f_n(t)$ for $n \geq 3$.

b) Calculate

$$\lim_{n\to\infty}\int_0^1 \frac{n + n^2 x}{(1+x)^n}\,dx\,.$$

State correctly the results and theorems you need to get to the solution.

Hint: b) To start, do the change of variable $t = nx$.
Solution: a) We have that

$$\left(1+\frac{t}{3}\right)^3 \le \left(1+\frac{t}{n}\right)^n \qquad \Leftrightarrow \qquad 3\,\log\left(1+\frac{t}{3}\right) \le n\log\left(1+\frac{t}{n}\right)$$

and if we define $F(t) = n\log\left(1+\frac{t}{n}\right) - 3\,\log\left(1+\frac{t}{3}\right)$ we have

$$F'(t) = \frac{t/3 - t/n}{(1+t/n)(1+t/3)} \ge 0 \implies \text{F is increasing.}$$

Hence,$F(t) \ge F(0) = 0$.
b) Doing the change of variable $t = nx$ we obtain that:

$$\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^n \frac{1+t}{(1+\frac{t}{n})^n}\,dt \le \int_0^n \frac{1+t}{(1+\frac{t}{3})^3}\,dt$$

using the part a) we obtain, for $n \ge 3$, that

$$\frac{1+t}{(1+\frac{t}{n})^n}\,\chi_{[0,n]}(t) \le \frac{1+t}{(1+\frac{t}{3})^3}\,\chi_{[0,\infty)}(t) \in L^1[0,\infty),$$

since

$$\int_0^\infty \frac{1+t}{(1+\frac{t}{3})^3}\,dt \le \int_0^1 (1+t)\,dt + \int_1^\infty \frac{1+t}{\frac{t^3}{27}}\,dt = \int_0^1 (1+t)\,dt + 27\int_1^\infty \frac{dt}{t^3} + 27\int_1^\infty \frac{dt}{t^2} < \infty\,.$$

Hence, since

$$\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t\,, \qquad t \ge 0\,,$$

using the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^\infty \lim_{n\to\infty}\frac{1+t}{(1+\frac{t}{n})^n}\,dt = \int_0^\infty (1+t)\,e^{-t}\,dt\,.$$

Now, the sequence of positive functions $G_N(t) = (1+t)\,e^{-t}\,\chi_{[0,N]}(t)$ is clearly increasing, and so, by the monotone convergence theorem:

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \int_0^1 \lim_{N\to\infty} G_N(t)\,dt = \lim_{N\to\infty}\int_0^N (1+t)\,e^{-t}\,dt\,.$$

But $(1+t)e^{-t}$ is continuous on $[0,N]$ for all N and therefore is Riemann-integrable in $[0,N]$. Hence,we can use Barrow's rule. By using integration by parts we can compute easily a primitive: $u = 1+t \implies du = dt$, $dv = e^{-t} \implies v = -e^{-t}$,

$$\int (1+t)\,e^{-t}\,dt = -(1+t)e^{-t} + \int e^{-t}\,dt = -(1+t)e^{-t} - e^{-t} = -(2+t)e^{-t}\,.$$

Finally,

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^2x}{(1+x)^n}\,dx = \lim_{N\to\infty}[-(2+t)e^{-t}]_{t=0}^{t=N} = 2 - \lim_{N\to\infty}\frac{2+N}{e^N} = 2 - \lim_{N\to\infty}\frac{1}{e^N} = 2-0 = 2\,,$$

where we have used L'Hopital rule.

Problem 2.2.17 Calculate $\lim_{n\to\infty}\int_0^1 \frac{n+n^4x^3}{(1+x)^n}\,dx\,.$

Solution: Proceeding as in the previous problem we begin proving that $\left(1+\frac{t}{5}\right)^5 \le \left(1+\frac{t}{n}\right)^n$ for $n\ge 5$ and $t\ge 0$, or equivalently that $5\,\log\left(1+\frac{t}{5}\right)\le n\log\left(1+\frac{t}{n}\right)$. To prove it, we define $F(t)=n\log\left(1+\frac{t}{n}\right)-5\,\log\left(1+\frac{t}{5}\right)$. We have that $F(t)\ge F(0)=0$ since

$$F'(t)=\frac{t/5-t/n}{(1+t/n)(1+t/5)}\ge 0 \implies \text{F is increasing.}$$

Now, we do the change of variable $t=nx$:

$$\int_0^1 \frac{n+n^4x^3}{(1+x)^n}\,dx=\int_0^n \frac{1+t^3}{(1+t/n)^n}\,dt\,.$$

Let $f_n(t)=\dfrac{1+t^3}{(1+t/n)^n}\,\chi_{[0,n]}(t)$. As $\lim_{n\to\infty}\left(1+\dfrac{t}{n}\right)^n=e^t$, we have that

$$\lim_{n\to\infty} f_n(t)=(1+t^3)\,e^{-t}\,\chi_{[0,\infty)}(t)\,.$$

On the other hand, $|f_n(t)|\le \frac{1+t^3}{(1+t/5)^5}\in L^1[0,\infty)$. Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^1 \frac{n+n^4x^3}{(1+x)^n}\,dx=\lim_{n\to\infty}\int_0^n \frac{1+t^3}{(1+t/n)^n}\,dt=\int_0^\infty (1+t^3)\,e^{-t}\,dt\,.$$

To compute this last integral, we apply the monotone convergence theorem and later we use integration by parts and L'Hopital rule:

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 \frac{n+n^4x^3}{(1+x)^n}\,dx &= \lim_{N\to\infty}\int_0^N (1+t^3)\,e^{-t}\,dt=\lim_{N\to\infty}\Big(-\big[(1+t^3)e^{-t}\big]_{t=0}^{t=N}+3\int_0^N t^2e^{-t}dt\Big)\\ &=1+3\lim_{N\to\infty}\Big(-\big[t^2e^{-t}\big]_{t=0}^{t=N}+2\int_0^N t\,e^{-t}dt\Big)=1+6\lim_{N\to\infty}\int_0^N t\,e^{-t}dt\\ &=1+6\lim_{N\to\infty}\Big(-\big[t\,e^{-t}\big]_{t=0}^{t=N}+\int_0^N e^{-t}dt\Big)=1+6=7\,.\end{aligned}$$

Problem 2.2.18 Prove that $\int_0^1 \frac{x}{1-x}\,\log\frac{1}{x}\,dx=\sum_{n=2}^\infty \frac{1}{n^2}\,.$

Hint: Use that $1/(1-x)=\sum_{n=0}^\infty x^n$ for $x\in(0,1)$ and then apply an adequate convergence theorem.

Solution: As $\frac{x}{1-x}=\sum_{n=1}^\infty x^n$ for $0\le x<1$, and $x^n\log(1/x)\ge 0$ on $[0,1)$, as a consequence of the monotone convergence theorem, the integral and series symbols commute:

$$\int_0^1 \frac{x}{1-x}\,\log\frac{1}{x}\,dx=\int_0^1\sum_{n=1}^\infty x^n\log\frac{1}{x}\,dx=\sum_{n=1}^\infty\int_0^1 x^n\log\frac{1}{x}\,dx\,.$$

Using again the monotone convergence theorem we obtain that:

$$\int_0^1 \frac{x}{1-x}\,\log\frac{1}{x}\,dx=\sum_{n=1}^\infty\lim_{N\to\infty}\int_{1/N}^1 x^n\log\frac{1}{x}\,dx\,.$$

But integrating by parts and using L'Hopital rule:

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 x^n\log\frac{1}{x}\,dx &= \lim_{\varepsilon\to 0^+}\Big(\Big[\frac{x^{n+1}}{n+1}\log\frac{1}{x}\Big]_{x=\varepsilon}^{x=1}+\int_\varepsilon^1\frac{x^n}{n+1}\,dx\Big)\\ &= \lim_{\varepsilon\to 0^+}\Big(-\frac{\varepsilon^{n+1}}{n+1}\log\frac{1}{\varepsilon}+\Big[\frac{x^{n+1}}{(n+1)^2}\Big]_{x=\varepsilon}^{x=1}\Big)\\ &= \lim_{\varepsilon\to 0^+}\frac{\varepsilon^{n+1}}{n+1}\log\varepsilon+\frac{1}{(n+1)^2}=\frac{1}{(n+1)^2}+\frac{1}{n+1}\lim_{\varepsilon\to 0^+}\frac{\log\varepsilon}{\varepsilon^{-(n+1)}}\\ &= \frac{1}{(n+1)^2}-\frac{1}{(n+1)^2}\lim_{\varepsilon\to 0^+}\frac{1/\varepsilon}{\varepsilon^{-(n+2)}}=\frac{1}{(n+1)^2}\Big(1-\lim_{\varepsilon\to 0^+}\varepsilon^{n+1}\Big)=\frac{1}{(n+1)^2}\,.\end{aligned}$$

Hence

$$\int_0^1\frac{x}{1-x}\,\log\frac{1}{x}\,dx=\sum_{n=1}^\infty\frac{1}{(n+1)^2}=\sum_{n=2}^\infty\frac{1}{n^2}\,.$$

Problem 2.2.19 Let $(X,\mathcal{P}(X),\mu)$ be a measure space with X countable, $X=\{x_n\}_{n=1}^\infty$, and μ the discrete measure defined as:

$$\mu(\{x_n\})=p_n\,,\qquad \mu(A)=\sum_{x_n\in A}p_n\,,\qquad (p_n\geq 0)\,.$$

Let $f:X\longrightarrow\mathbb{C}$ be a complex function.

a) Prove that if $f\geq 0$, then $\displaystyle\int_X f\,d\mu=\sum_{n=1}^\infty f(x_n)\,p_n$.

b) Prove that $f\in L^1(\mu)$ if and only if $\sum_{n=1}^\infty|f(x_n)|\,p_n<\infty$, and in this case,

$$\int_X f\,d\mu=\sum_{n=1}^\infty f(x_n)\,p_n\,.$$

Hints: a) $f=\sum_{n=1}^\infty f(x_n)\,\chi_{\{x_n\}}$. b) Decompose $f=u+iv$ and $u=u^+-u^-$, $v=v^+-v^-$.

Solution: a) It is clear that $f=\sum_{n=1}^\infty f(x_n)\,\chi_{\{x_n\}}$. Hence, as $f\geq 0$, as a consequence of monotone convergence theorem, the integral and series symbols commute:

$$\begin{aligned}\int_X f\,d\mu &= \int_X\sum_{n=1}^\infty f(x_n)\,\chi_{\{x_n\}}(x)\,d\mu(x)=\sum_{n=1}^\infty\int_X f(x_n)\,\chi_{\{x_n\}}(x)\,d\mu(x)\\ &= \sum_{n=1}^\infty f(x_n)\,\mu(\{x_n\})=\sum_{n=1}^\infty f(x_n)\,p_n\,.\end{aligned}$$

b) By part a) we have: $f\in L^1(\mu)\iff\int_X|f|\,d\mu<\infty\iff\sum_{n=1}^\infty|f(x_n)|\,p_n<\infty$. Besides, in this case:

b.1) If $f:X\longrightarrow\mathbb{R}$, then $f=f^+-f^-$ with $f^+,f^-\geq 0$ and by a):

$$\begin{aligned}\int_X f\,d\mu &= \int_X f^+\,d\mu-\int_X f^-\,d\mu=\sum_{n=1}^\infty f^+(x_n)\,p_n-\sum_{n=1}^\infty f^-(x_n)\,p_n\\ &= \sum_{n=1}^\infty(f^+(x_n)-f^-(x_n))\,p_n=\sum_{n=1}^\infty f(x_n)\,p_n\,.\end{aligned}$$

b.2) If $f : X \longrightarrow \mathbb{C}$, then $f = u + iv$ with $u, v : X \longrightarrow \mathbb{R}$ and by b.1):

$$\int_X f\,d\mu = \int_X u\,d\mu + i\int_X v\,d\mu = \sum_{n=1}^{\infty} u(x_n)\,p_n + i\sum_{n=1}^{\infty} v(x_n)\,p_n$$
$$= \sum_{n=1}^{\infty}\big(u(x_n) + iv(x_n)\big)\,p_n = \sum_{n=1}^{\infty} f(x_n)\,p_n\,.$$

Problem 2.2.19 Calculate $\lim\limits_{n\to\infty} n\sum\limits_{i=1}^{\infty}\sin\dfrac{2^{-i}}{n}$.

Hint: Consider and adequate measure space and apply a convergence theorem.

Solution: Here the measure space is $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is the counting measure, i.e. the measure considered in the previous problem with $p_n = 1$ for all n.

Let $\varphi_n(i) = n\sin(2^{-i}/n)$. Since $\sin x \le x$ for $x \in [0, \pi/2]$ we have: $|\varphi_n(i)| \le 2^{-i}$, for all $n \in \mathbb{N}$. Also $2^{-i} \in L^1(\mu)$ since $\sum_{i=1}^{\infty} 2^{-i} < \infty$. Hence, by the dominated convergence theorem:

$$\lim_{n\to\infty} n\sum_{i=1}^{\infty}\sin\frac{2^{-i}}{n} = \lim_{n\to\infty}\sum_{i=1}^{\infty}\varphi_n(i) = \sum_{i=1}^{\infty}\lim_{n\to\infty}\varphi_n(i)$$
$$= \sum_{i=1}^{\infty}\lim_{n\to\infty} n\sin\frac{2^{-i}}{n} = \sum_{i=1}^{\infty} 2^{-i} = \frac{1}{1-\frac{1}{2}} - 1 = 2 - 1 = 1\,.$$

Problem 2.2.21* Let $(X, \mathcal{A}, \mu)$ be a measure space and $\Phi : X \longrightarrow Y$ be a mapping. Let us consider the image measure space $(Y, \mathcal{B}, \nu)$ by Φ ($\mathcal{B} = \Phi(\mathcal{A})$ and $\nu = \mu \circ \Phi^{-1}$). Let $f : Y \longrightarrow \mathbb{C}$ be a function. Prove that

a) f is $\mathcal{B}$-measurable if and only if $f \circ \Phi$ is $\mathcal{A}$-measurable .

b) If $f \ge 0$ is $\mathcal{B}$-measurable, then $\int_Y f\,d\nu = \int_X (f\circ\Phi)\,d\mu$.

c) If f is $\mathcal{B}$-measurable, then $f \in L^1(\nu)$ if and only if $f \circ \Phi \in L^1(\mu)$, and in this case

$$\int_Y f\,d\nu = \int_X (f\circ\Phi)\,d\mu.$$

d) Let $\Phi(x, y) = x^2 y$ be defined on the square $Q = [0, 1] \times [0, 1]$ in the plane, and let m be two-dimensional Lebesgue measure on Q. If μ is the image measure of m by Φ, evaluate the integral $\int_{-\infty}^{\infty} t^2\,d\mu(t)$.

Hints: a) Use the definition of $\mathcal{A}$. b) Prove it first for simple functions and then approximate any $f \ge 0$ by simple functions and apply monotone convergence. c) Decompose $f = u + iv$ and $u = u^+ - u^-$, $v = v^+ - v^-$. d) Apply c).

Solution: a) Let B be a borelian subset in $\mathbb{C}$. Then, by definition,

$$f^{-1}(B) \in \mathcal{B} \quad\Longleftrightarrow\quad \Phi^{-1}(f^{-1}(B)) = (f\circ\Phi)^{-1}(B) \in \mathcal{A}\,.$$

b) First, if $s = \chi_B$ with $B \in \mathcal{B}$, then

$$\int_Y s\,d\nu = \nu(B) = \mu(\Phi^{-1}(B)) = \int_X \chi_{\Phi^{-1}(B)}\,d\mu = \int_X (\chi_B\circ\Phi)\,d\mu = \int_X (s\circ\Phi)\,d\mu\,. \qquad (2)$$

Secondly, if $s = \sum_{j=1}^{n} c_j \chi_{B_j}$ is a simple function, then using (2):

$$\int_Y s\, d\nu = \int_Y \Big(\sum_{j=1}^{n} c_j \chi_{B_j} \Big)\, d\nu = \sum_{j=1}^{n} c_j \int_Y \chi_{B_j}\, d\nu$$
$$= \sum_{j=1}^{n} c_j \int_X (\chi_{B_j} \circ \Phi)\, d\mu = \int_X \Big(\sum_{j=1}^{n} c_j (\chi_{B_j} \circ \Phi) \Big)\, d\nu = \int_X (s \circ \Phi)\, d\mu\,.$$

Finally, if $f \geq 0$ is $\mathcal{B}$-measurable, then let $\{s_n\}_{n=1}^{\infty}$ be a sequence of positive simple functions in Y such that

$$0 \leq s_1 \leq \cdots \leq s_n \cdots \nearrow f\,, \qquad \text{as } n \to \infty\,.$$

But $s_n \circ \Phi$ are positive simple functions in X such that

$$0 \leq s_1 \circ \Phi \leq \cdots \leq s_n \circ \Phi \cdots \nearrow f \circ \Phi\,, \qquad \text{as } n \to \infty\,.$$

Using now twice the monotone convergence theorem:

$$\int_Y f\, d\nu = \lim_{n\to\infty} \int_Y s_n\, d\nu = \lim_{n\to\infty} \int_X (s_n \circ \Phi) = \int_X (f \circ \Phi)\, d\mu\,.$$

c) If f is $\mathcal{B}$-measurable, then by part b):

$$f \in L^1(\mu) \iff \int_Y |f|\, d\nu < \infty \iff \int_X (|f| \circ \Phi)\, d\mu = \int_X |f(\Phi)|\, d\mu < \infty \iff f \circ \Phi \in L^1(\mu).$$

Besides, in this case, if If $f : X \longrightarrow \mathbb{R}$, then $f = f^+ - f^-$ with $f^+, f^- \geq 0$ and by b)

$$\int_Y f\, d\nu = \int_Y (f^+ - f^-)\, d\nu = \int_Y f^+\, d\nu - \int_Y f^-\, d\nu$$
$$= \int_X (f^+ \circ \Phi)\, d\mu - \int_X (f^- \circ \Phi)\, d\mu = \int_X \big((f^+ - f^-) \circ \Phi \big)\, d\mu = \int_X (f \circ \Phi)\, d\mu\,.$$

Finally, if $f : X \longrightarrow \mathbb{C}$, then $f = u + iv$ with $u, v : X \longrightarrow \mathbb{R}$ and by the previous identity:

$$\int_Y f\, d\nu = \int_Y (u + iv)\, d\nu = \int_Y u\, d\nu + i \int_Y v\, d\nu$$
$$= \int_X (u \circ \Phi)\, d\mu + i \int_X (v \circ \Phi)\, d\mu = \int_X \big((u + iv) \circ \Phi \big)\, d\mu = \int_X (f \circ \Phi)\, d\mu\,.$$

d) Using part b) and applying Fubini's theorem we get

$$\int_{\mathbb{R}} t^2\, d\mu(t) = \int_{[0,1]\times[0,1]} (t^2 \circ \Phi)\, dx\, dy = \int_0^1 \int_0^1 (\Phi(x,y))^2\, dx\, dy = \int_0^1 \int_0^1 (x^2 y)^2\, dx\, dy$$
$$= \Big(\int_0^1 x^4 dx \Big) \Big(\int_0^1 y^2 dy \Big) = \Big[\frac{x^5}{5} \Big]_{x=0}^{x=1} \Big[\frac{y^3}{3} \Big]_{y=0}^{y=1} = \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}\,.$$

Problem 2.2.22 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\rho : X \longrightarrow [0, \infty]$ be a measurable function. Let us consider the measure defined by the density ρ:

$$\nu(A) = \int_A \rho \, d\mu \,, \qquad A \in \mathcal{A}.$$

Prove that

a) If $f \geq 0$ is measurable, then $\displaystyle\int_X f \, d\nu = \int_X f \rho \, d\mu$.

b) If f is measurable, then: $f \in L^1(\nu)$ if and only if $\displaystyle\int_X |f| \rho \, d\mu < \infty$, and in this case

$$\int_X f \, d\nu = \int_X f \rho \, d\mu.$$

Hints: a) This is the exercise 2.1.3. b) Decompose $f = u + iv$ and $u = u^+ - u^-$, $v = v^+ - v^-$.
Solution: a) This is the exercise 2.1.3. b) If If $f : X \longrightarrow \mathbb{R}$, then $f = f^+ - f^-$ with $f^+, f^- \geq 0$ and by b)

$$\begin{aligned}\int_X f \, d\nu &= \int_X (f^+ - f^-) \, d\nu = \int_X f^+ \, d\nu - \int_X f^- \, d\nu \\ &= \int_X f^+ \rho \, d\mu - \int_X f^- \rho \, d\mu = \int_X (f^+ - f^-) \, \rho \, d\mu = \int_X f \, \rho \, d\mu \,.\end{aligned}$$

Finally, if $f : X \longrightarrow \mathbb{C}$, then $f = u + iv$ with $u, v : X \longrightarrow \mathbb{R}$ and by the previous identity:

$$\begin{aligned}\int_X f \, d\nu &= \int_X (u + iv) \, d\nu = \int_X u \, d\nu + i \int_X v \, d\nu \\ &= \int_X u \, \rho \, d\mu + i \int_X v \rho \, d\mu = \int_X (u + iv) \, \rho \, d\mu = \int_X f \, \rho \, d\mu \,.\end{aligned}$$

Problem 2.2.23 Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $dm = dx \, dy$ be the Lebesgue measure on X. Let $\Phi : X \longrightarrow \mathbb{R}$ be the function given by $\Phi(x, y) = \log(x^2 + y^2)$ and let μ be the image measure of dm by Φ.

a) Calculate the value of $\mu([0, 1])$.

b) Prove that μ has the form $d\mu = F(t) \, dt$ and find $F(t)$ explicitly.

Hints: a) $\mu([0, 1]) = m(\{(x, y) : 0 \leq \log(x^2 + y^2) \leq 1\})$. b) Calculate $\int_{\mathbb{R}} f(t) d\mu(t)$ for any $f \in L^1(\mu)$.
Solution: a) By definition of image measure we have

$$\begin{aligned}\mu([0, 1]) &= m(\Phi^{-1}([0, 1]) = m\big(\{(x, y) : \Phi(x, y) \in [0, 1]\}\big) = m\big(\{(x, y) : 0 \leq \log(x^2 + y^2) \leq 1\}\big) \\ &= m\big(\{(x, y) : 1 \leq x^2 + y^2 \leq e\}\big) = \pi((\sqrt{e}\,)^2 - 1) = \pi(e - 1) \,.\end{aligned}$$

b) Let $f \in L^1(\mu)$. Then, using polar coordinates,

$$\begin{aligned}\int_{\mathbb{R}} f(y) \, d\mu(y) &= \iint_X (f \circ \Phi)(x, y) \, dx \, dy = \iint_X f\big(\log(x^2 + y^2)\big) \, dx \, dy = \\ &= \int_0^{2\pi} \int_0^{\infty} f(\log r^2) \, r \, dr \, d\theta = 2\pi \int_0^{\infty} f(\log r^2) \, r \, dr \,,\end{aligned}$$

and doing the change of variable $y = \log r^2$ we obtain

$$\int_{\mathbb{R}} f(y)\, d\mu(y) = 2\pi \int_{\mathbb{R}} f(y)\, e^{y/2}\, \frac{1}{2}\, e^{y/2}\, dy = \int_{\mathbb{R}} f(y)\, \pi\, e^{y}\, dy\,.$$

Hence, $d\mu = F(y)\, dy$ where $F(y) = \pi\, e^y$.

Problem 2.2.24

a) Let $f : \mathbb{R} \to [0, \infty]$ be an integrable function on $\mathbb{R}$ and such that $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Prove that $F(x) = \int_{-\infty}^{x} f(y) dy$ is a probability distribution function and that besides F is continuous (f is called the density function).

b) Prove that the Borel-Stieltjes measure with distribution function F coincides with the measure defined with the density function f: $\nu_f(A) = \int_A f(x)\, dx$.

c) Calculate $F(x)$ if

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1]\,, \\ 0 & \text{otherwise.} \end{cases}$$

Hints: a) F is increasing because $f \geq 0$ and is continuous by the dominated convergence theorem. b) The Borel-Stieltjes measure μ_F coincides with the density measure ν_f by the Caratheodory-Hopf's extension theorem since for semi-intervals $[a, b)$ we have: $\mu_F([a, b)) = F(b) - F(a) = \int_a^b f(x)\, dx = \nu_f([a, b))$. Observe that $\mu_F(\{a\})=0$ for all $a \in \mathbb{R}$ since F is continuous. c) $F(x) = 0$, if $x \leq 0$, $F(x) = x$ if $x \in [0, 1]$ and $F(x) = 1$ if $x \geq 1$.

a) As $f \geq 0$, then F is increasing since for $x_1 \leq x_2$ we have that

$$F(x_2) - F(x_1) = \int_{-\infty}^{x_2} f(y)\, dy - \int_{-\infty}^{x_1} f(y)\, dy = \int_{x_1}^{x_2} f(y)\, dy \geq 0\,.$$

Also, if $x_n \to x_0$ as $n \to \infty$ we have that

$$F(x_n) = \int_{-\infty}^{x_n} f(y)\, dy = \int_{\mathbb{R}} f(y)\, \chi_{(-\infty, x_n)}(y)\, dy$$

and $|f\, \chi_{(-\infty,x_n)}| \leq f \in L^1(\mathbb{R})$. Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} F(x_n) = \int_{\mathbb{R}} \Big(\lim_{n\to\infty} f(y)\, \chi_{(-\infty,x_n)}(y) \Big)\, dy = \int_{\mathbb{R}} f(y)\, \chi_{(-\infty,x_0)}(y)\, dy = F(x_0)\,.$$

Hence, F is continuous.

F is the distribution function of the probability measure given by the density f: $\nu_f(A) = \int_A f(y)\, dy$ for any borelian A in $\mathbb{R}$.

b) For all semiopen interval $[a, b)$ we have, since F is continuous, that

$$\mu_F([a, b)) = F(b) - F(a) = \int_a^b f(y)\, dy = \int_{[a,b)} f(y)\, dy = \nu_f([\alpha, b))\,.$$

Hence, by Caratheodory-Hopf's extension theorem, $\mu_F = \nu_f$.

c) We have

$$F(x) = \int_{-\infty}^{x} f(y)\, dy = \begin{cases} 0\,, & \text{if } x \leq 0\,, \\ \int_0^x 1\, dy = x\,, & \text{if } 0 \leq x \leq 1\,, \\ \int_0^1 1\, dy = 1\,, & \text{if } 1 \leq x\,. \end{cases}$$

Problem 2.2.25* Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing derivable function with bounded derivative g' on each compact set. Let us consider the Borel-Stieltjes measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m_g)$. Prove that $m_g = g'dm$, that is to say that the Borel-Stieltjes measure m_g coincides with the measure defined by the density g' and therefore for all $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in L^1(m_g)$, we have

$$\int_{\mathbb{R}} f \, dm_g = \int_{\mathbb{R}} f g' \, dm = \int_{\mathbb{R}} f(t) \, g'(t) \, dt \, .$$

Hint: Use the Caratheodory-Hopf extension theorem and that $\int_a^b g' \, dm = g(b) - g(a)$. This is trivial if g' is continuous by Barrow's rule, but for g' only bounded we must use an approximation argument: let $g_n(t) = (f(t + h_n) - f(t)/h_n$. Then $g_n \longrightarrow g'$ for all $t \in [a, b)$. Use dominated convergence to conclude that $\int_a^c g' \, dm = g(c) - g(a)$ for all $c \in [a, b)$. Finally use monotone convergence, since $[a, b) = \cup_n [a, c_n]$ with $c_n \nearrow b$ as $n \to \infty$.

Problem 2.2.26* Let us consider the Lebesgue measure space $(\mathbb{R}^n, \mathcal{M}, m)$, where $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets and m is Lebesgue measure. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function. Prove that

a) If $f \geq 0$ or if $f \in L^1(m)$, then

a.1) $\displaystyle\int_{\mathbb{R}^n} f(a + x) \, dx = \int_{\mathbb{R}^n} f(x) \, dx \, .$

a.2) $\displaystyle\int_{\mathbb{R}^n} f(T(x)) \, dx = \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(x) \, dx$, for all $T \in GL(n)$.

a.3) More generally, $\displaystyle\int_{A} f(T(x)) \, dx = \frac{1}{|\det T|} \int_{T(A)} f(x) \, dx$, for all $T \in GL(n)$ and $A \in \mathcal{M}$.

b) If $\Phi : \mathbb{R} \longrightarrow [0, \infty]$ is a Borel function, then

$$\int_{\mathbb{R}^n} \Phi(\|x\|) \, dx = n\Omega_n \int_0^\infty \Phi(r) \, r^{n-1} \, dr \, , \qquad \text{where } \Omega_n = m(\{x \in \mathbb{R}^n : \ \|x\| \leq 1\}) \, .$$

c) Let $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$. Then

$$\int_{B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha < n \qquad \text{and} \qquad \int_{\mathbb{R}^n \setminus B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha > n \, .$$

Hints: Let $\mu = T(m)$ be the image measure of m under T: a.1) If $T(x) = a+x$, then $\mu(A) = m(A)$ since m is translation-invariant. a.2) $\mu(A) = m(T^{-1}(A)) = |\det T^{-1}| \, m(A)$. This fact is easy for semi-intervals $[a_1, b_1) \times \cdots \times [a_n, b_n)$ and so it is a consequence of Caratheodory-Hopf extension theorem. a.3) It follows from a.2) and the fact that, as T is bijective, we have $\chi_{T(A)} \circ T = \chi_A$. b) Let $\nu = \| \cdot \| \circ m$ be the image measure under $\| \cdot \|$: then prove that $\nu[a, b) = \Omega_n (b^n - a^n)$ and as $g(t) = \Omega_n t^n$ is increasing and continuous, conclude from Exercise 2 that $\nu = g' \, dm = n\Omega_n t^{n-1} dt$. c) Apply part b).

Problem 2.2.27* Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue-integrable function. Evaluate

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x - n) \, \frac{x}{1 + |x|} \, dx \, .$$

Hint: Apply the change of variables $y = x + n$ and divide the integral in two parts: one on the interval $(-\infty, -n)$ and the other one on $(-n, \infty)$. Apply Lebesgue dominated convergence theorem to prove that the first integral converges to 0 and the second one to $\int_{-\infty}^{\infty} f(x)\,dx$.
Solution: $\int_{-\infty}^{\infty} f(x)\,dx$.