uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.5: L^p-spaces

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

2 Integration Theory

2.5. L^p-spaces

Problem 2.5.1 Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be functions such that

$$\varphi_i \in L^{p_i}(X, \mathcal{A}, \mu), \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \leq 1.$$

Then $\varphi_1\varphi_2\cdots\varphi_k \in L^p(X, \mathcal{A}, \mu)$ and $\|\varphi_1\varphi_2\cdots\varphi_k\|_p \leq \|\varphi_1\|_{p_1}\|\varphi_2\|_{p_2}\cdots\|\varphi_k\|_{p_k}$.

Hint: If $a_1, \cdots, a_k \geq 0$ and $\lambda_1 + \cdots \lambda_k = 1$, then $a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_k^{\lambda_k} \leq \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k$.

Solution: If $\|\varphi_i\|_{p_i} = 0$ for some i, then $\varphi_i = 0$ a.e. and so the inequality is obvious. The result is also trivial if $\|\varphi_i\|_{p_i} = \infty$ for some i. Hence, we can assume that $0 < \|\varphi_i\|_{p_i} < \infty$ for all $i \in \{1, \ldots, k\}$. Also, by homogeneity it suffices to prove that

$$\|\varphi_1\varphi_2\cdots\varphi_k\|_p \leq 1, \qquad \text{if } \|\varphi_i\|_{p_i} = 1, \quad \text{for } i = 1, \ldots, k.$$

From the convexity of the exponential function and using Jensen's inequality it is easy to check that

$$a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_k^{\lambda_k} \leq \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k, \qquad \text{if } a_1, \ldots, a_k \geq 0. \tag{1}$$

Let us take $a_i = |\varphi_i(x)|^{p_i}$ and $\lambda_i = p/p_i$, $i = 1, \ldots, k$. Then, using (1), we have

$$\begin{aligned} |\varphi_1(x)\varphi_2(x)\cdots\varphi_k(x)|^p &= |\varphi_1(x)|^{\lambda_1 p_1}|\varphi_2(x)|^{\lambda_2 p_2}\cdots|\varphi_k(x)|^{\lambda_k p_k} \\ &\leq \lambda_1\,|\varphi_1(x)|^{p_1} + \lambda_2\,|\varphi_2(x)|^{p_2} + \cdots + \lambda_k\,|\varphi_k(x)|^{p_k} \\ &= \frac{p}{p_1}\,|\varphi_1(x)|^{p_1} + \frac{p}{p_2}\,|\varphi_2(x)|^{p_2} + \cdots + \frac{p}{p_k}\,|\varphi_k(x)|^{p_k} \end{aligned}$$

and, as $\|\varphi_i\|_{p_i} = 1$, we obtain integrating that

$$\int_X |\varphi_1(x)\varphi_2(x)\cdots\varphi_k(x)|^p\,d\mu \leq \sum_{i=1}^k \frac{p}{p_i}\int_X |\varphi_i(x)|^{p_i}d\mu = \sum_{i=1}^k \frac{p}{p_i} = 1.$$

Problem 2.5.2 Let $0 < p < r < q \leq \infty$ and let $\varphi \in L^p(X, \mathcal{A}, \mu) \cap L^q(X, \mathcal{A}, \mu)$.

a) Prove that $\varphi \in L^r(X, \mathcal{A}, \mu)$ and

$$\|\varphi\|_r \leq \|\varphi\|_p^\theta \|\varphi\|_q^{1-\theta}, \qquad \text{where } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

b) Prove also that $L^r(\mu) \subset L^p(\mu) + L^q(\mu)$.

c) Prove that $\lim_{r\to\infty} \|\varphi\|_r = \|\varphi\|_\infty$.

Hints: a) If $q = \infty$, then $|\varphi|^r = |\varphi|^{r-p}|\varphi|^p \leq \|\varphi\|_\infty^{r-p}|\varphi|^p$ and $\frac{1}{r} = \frac{\theta}{p}$. If $q < \infty$, then $\frac{p}{\theta r}$ and $\frac{q}{(1-\theta)r}$ are conjugate exponents and $|\varphi|^r = |\varphi|^{\theta r}|\varphi|^{(1-\theta)r}$. Apply Hölder's inequality. b) If $A = \{x \in X :\ |\varphi(x)| \leq 1\}$, then $\varphi = \varphi\chi_A + \varphi\chi_{A^c}$. c) By letting $r \to \infty$ in $\|\varphi\|_r \leq \|\varphi\|_p^\theta\|\varphi\|_\infty^{1-\theta}$ deduce that $\limsup_{r\to\infty} \|\varphi\|_r \leq \|\varphi\|_\infty$. Also, we can suppose that $\|\varphi\|_\infty > a > 0$. Use Markov's inequality to deduce that $\|\varphi\|_r \geq a\,\mu(\{x : |\varphi(x)| > a\})^{1/r}$ and by letting $r \to \infty$ and $a \to \|\varphi\|_\infty$ deduce that $\liminf_{r\to\infty} \|\varphi\|_r \geq \|\varphi\|_\infty$.

Problem 2.5.3 Let $(X, \mathcal{A}, \mu)$ be a measure space. For some measures the relation $p < q$ implies $L^p \subset L^q$. For others the relationship is reversed and there are some measures for which L^p does no contain L^q for $p \neq q$. Give examples of these situations:

a) If $\mu(X) < \infty$ and $1 \leq p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq \|f\|_q \, \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

b) If $0 < p < q \leq \infty$, then $\ell^p \subset \ell^q$ and $\|x_n\|_q \leq \|x_n\|_p$.

c) Show that $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), m) \not\subseteq L^q(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ for $p \neq q$.

Hints: a) Use Hölder's inequality. b) Use part *a*) of problem 2.5.2. c) Consider the function $f(x) = |x(\log^2|x| + 1)|^{-1/p}$.

Solution: a) If $q = \infty$, it is obvious. If $q < \infty$, we use Hölder's inequality with the conjugate exponents q/p and $(q/p)' = q/(q-p)$:

$$\|f\|_p^p = \int_X |f|^p \cdot 1 \, d\mu \leq \big\||f|^p\big\|_{q/p} \|1\|_{(q/p)'} = \Big(\int_X |f|^q d\mu\Big)^{p/q} \mu(X)^{(q-p)/q}$$

and so $\|f\|_p \leq \|f\|_q \, \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

b) Obviously, $\|x_n\|_\infty = \sup_n |x_n| \implies \|x_n\|_\infty^p = (\sup_n |x_n|)^p \leq \sum_n |x_n|^p = \|x_n\|_p^p$ and so $\|x_n\|_\infty \leq \|x_n\|_p$. The case $q < \infty$ follows from problem 2.5.2 and the inequality just proved $\|x_n\|_\infty \leq \|x_n\|_p$:

$$\|x_n\|_q \leq \|x_n\|_p^{p/q} \|x_n\|_\infty^{1-p/q} \leq \|x_n\|_p^{p/q} \|x_n\|_p^{1-p/q} = \|x_n\|_p \,.$$

c) Let $f(x) = \dfrac{1}{|x(\log^2|x| + 1)|^{1/p}}$ and let us assume that $p < q$. Then

$$\int_{\mathbb{R}} |f(x)|^p \, dx = \int_{\mathbb{R}} \frac{dx}{|x| \, (\log^2|x| + 1)} = 2 \int_0^\infty \frac{dx}{|x| \, (\log^2|x| + 1)} < \infty$$

since

$$\int_0^\infty \frac{dx}{|x| \, (\log^2|x| + 1)} \leq \int_0^{1/2} \frac{dx}{x \log^2 x} + \int_{1/2}^2 \frac{dx}{x \log^2 x + 1} + \int_2^\infty \frac{dx}{x \log^2 x} < \infty$$

because using the monotone convergence theorem we have

$$\int_0^{1/2} \frac{dx}{x \log^2 x} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{1/2} \frac{dx}{x \log^2 x} = \lim_{\varepsilon \to 0^+} \Big[\frac{-1}{\log x}\Big]_{x=\varepsilon}^{x=1/2} = \lim_{\varepsilon \to 0^+} \frac{1}{\log \varepsilon} - \frac{1}{\log \frac{1}{2}} = \frac{1}{\log 2} < \infty \,,$$

$$\int_{1/2}^2 \frac{dx}{x \log^2 x + 1} < \infty \,, \qquad \text{since } \frac{1}{x \log^2 x + 1} \text{ is continuous in } [1/2, 2] \,,$$

$$\int_2^\infty \frac{dx}{x \log^2 x} = \lim_{n \to \infty} \int_2^N \frac{dx}{x \log^2 x} = \lim_{N \to \infty} \Big[\frac{-1}{\log x}\Big]_{x=2}^{x=N} = \lim_{N \to \infty} \frac{1}{\log 2} - \frac{1}{\log N} = \frac{1}{\log 2} < \infty \,.$$

We also have

$$\int_{\mathbb{R}} |f(x)|^q \, dx = \int_{\mathbb{R}} \frac{dx}{|x|^{q/p} \, (\log^2|x| + 1)^{q/p}} = 2 \int_0^\infty \frac{dx}{x^{q/p} \, (\log^2 x + 1)^{q/p}} = \infty$$

since, $\lim_{x\to 0^+} x^\delta \log x = 0$ for all $\delta > 0$, and so $(\log^2 x + 1)^{q/p} \le C(\delta)/x^\delta$ in $(0, 1/2]$, and using the monotone convergence theorem we get that

$$\int_0^{1/2} \frac{dx}{x^{q/p}\,(\log^2 x + 1)^{q/p}} \ge C \int_0^{1/2} \frac{dx}{x^{q/p-\delta}} = C \lim_{\varepsilon\to 0^+} \int_\varepsilon^{1/2} \frac{dx}{x^{q/p-\delta}} = C \lim_{\varepsilon\to 0^+} \Big[\frac{x^{1+\delta-q/p}}{1+\delta-q/p}\Big]_{x=\varepsilon}^{x=1/2}$$
$$= \frac{C}{q/p-1-\delta} \lim_{\varepsilon\to 0^+} \Big(\frac{1}{\varepsilon^{q/p-\delta-1}} - 2^{q/p-\delta-1}\Big) = \infty$$

if we choose $\delta > 0$ small enough so that $1 + \delta < q/p$.
Let us assume now that $q < p$. Then the same function f verifies that $f \notin L^q$, since $\lim_{x\to\infty} \log x/x^\delta = 0$ for all $\delta > 0$ and so $(\log^2 x + 1)^{q/p} \le C(\delta)\, x^\delta$ in $(1, \infty)$. Hence, using again the monotone convergence theorem we have that

$$\int_1^{\infty} \frac{dx}{x^{q/p}\,(\log^2 x + 1)^{q/p}} \ge C \int_1^{\infty} \frac{dx}{x^{q/p+\delta}} = C \lim_{N\to\infty} \Big[\frac{x^{1-\delta-q/p}}{1-\delta-q/p}\Big]_{x=1}^{x=N}$$
$$= \frac{C}{1-\delta-q/p} \lim_{N\to\infty} \big(N^{1-\delta-q/p} - 1\big) = \infty$$

if we choose $\delta > 0$ small enough so that $q/p + \delta < 1$.

Problem 2.5.4 Let $(X, \mathcal{A}, \mu)$ be a measure space.

i) Prove that Hölder's inequality holds for the exponents $p = 1$ and $q = \infty$: If f and g are measurable functions on X, then $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$.

ii) If $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, prove that $\|fg\|_1 = \|f\|_1\|g\|_\infty$ iff $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

iii) Prove that if $f \in L^p(\mu)$ and $g \in L^\infty(\mu)$, then $fg \in L^p(\mu)$ and $\|fg\|_p \le \|f\|_p\|g\|_\infty$. When equality holds in this inequality?

iv) Prove that $\|\cdot\|_\infty$ is a norm on $L^\infty(\mu)$.

v) Prove that if $\mu(X) < \infty$ and $f \in L^\infty(\mu)$, then $f \in \cap_{p\ge 1} L^p(\mu)$. Prove that the reverse statement is false.

vi) Let $f \in L^\infty(\mu)$ and $\{f_n\}$ be a sequence in $L^\infty(\mu)$. Prove that $\|f_n - f\|_\infty \to 0$ if and only if there exists $E \in \mathcal{A}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

vii) The simple functions are dense in L^∞ if $\mu(X) < \infty$: Each $f \in L^\infty$ can be approximated by a sequence of simple functions $\{s_n\} \subset L^\infty(\mu)$.

Hint: v) Consider the function $f(x) = \log x$ on $X = (0, 1]$.
Solution: i) $\|fg\|_1 = \int_X |fg|\, d\mu \le \int_X |f|\, \|g\|_\infty\, d\mu = \|g\|_\infty \|f\|_1$.
ii) $\|fg\|_1 = \|f\|_1\|g\|_\infty \iff \int_X |f|(\|g\|_\infty - |g|)d\mu = 0 \iff \|g\|_\infty = |g(x)|$ a.e. on the set where $f(x) \neq 0$, as $\|g\|_\infty - |g| \ge 0$ a.e.
iii) $\|fg\|_p^p = \int_X |fg|^p d\mu \le \int_X |f|^p \|g\|_\infty^p d\mu \le \|g\|_\infty^p \|f\|_p^p$. Equality holds if and only if $\int_X |f|^p(\|g\|_\infty^p - |g|^p)d\mu = 0 \iff |g| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$, as $\|g\|_\infty^p - |g|^p \ge 0$ a.e.
iv) a) $\|g\|_\infty = 0 \iff |g| = 0$ a.e. $\iff g = 0$ a.e. $\iff g = 0 \in L^\infty(\mu)$. b) $\|\lambda g\|_\infty = |\lambda|\|g\|_\infty$ because $|\lambda\, g(x)| = |\lambda|\, |g(x)| \le |\lambda|\, \|g\|_\infty$ a.e. and $\mu(\{x : |\lambda|\, |g(x)| > |\lambda|\, \|g\|_\infty\}) = \mu(\{x : |g(x)| > \|g\|_\infty\})$. c) $|f + g| \le |f| + |g| \implies \|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

v) $\|f\|_p^p = \int_X |f|^p d\mu \leq \|f\|_\infty \int_X d\mu = \|f\|_\infty \mu(X) < \infty$, $\forall\, 1 \leq p < \infty$. But, $f(x) = \log x \notin L^\infty(0,1]$ but $f \in L^p(0,1]$ for all $1 \leq p < \infty$: $\int_0^1 |\log x|^p dx \leq C\int_0^\delta \frac{1}{x^{\varepsilon p}}dx + \int_\delta^1 |\log x|^p dx < \infty$ choosing ε so that $\varepsilon p < 1$.

vi) ($\Leftarrow$) As $f_n \to f$ uniformly on E, given $\varepsilon > 0$, $\exists\, N = N(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall\, n > N$, $\forall\, x \in E$ and therefore, as $\mu(E^c) = 0$, we have that $\|f_n - f\|_\infty \leq \varepsilon$, $\forall\, n > N$. Hence, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.
($\Rightarrow$) If $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, then $\forall\, k \in \mathbb{N}$, $\exists\, N = N(k)$ such that $\|f_n - f\|_\infty < 1/k$, $\forall\, n > N(k)$. Hence, the set $E_{k,n}^c := \{x \in X : \|f_n - f\|_\infty \geq 1/k\}$ has $\mu(E_{k,n}^c) = 0$. Hence, the set $E := (\cup_{k,n} E_{k,n}^c)^c = \cap_{k,n} E_{k,n}$ verifies $\mu(E^c) = \mu(\cup_{k,n} E_{k,n}^c) = 0$ and $|f_n(x) - f(x)| < 1/k$, $\forall\, n > N(k)$, $\forall\, x \in E$. Therefore, $f_n - f \to 0$ as $n \to \infty$ uniformly on E, and $\mu(E^c) = 0$.

vii) If $f \in L^\infty(\mu)$, then we can choose a bounded representative of f, i.e. we can assume that $|f| \leq \|f\|_\infty$, $\forall\, x \in X$. Also, it is enough to prove it for $f \geq 0$. In this case, there exists a sequence $\{s_n\}$ of simple functions such that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \cdots \nearrow f$, as $n \to \infty$. But then $\|s_n - f\|_\infty \to 0$ as $n \to \infty$.

Problem 2.5.5 Let $1 \leq p < \infty$.

a) Show that if $\varphi \in L^p(\mathbb{R}^N)$ and φ is uniformly continuous, then $\lim_{|x|\to\infty} \varphi(x) = 0$.
b) Show that this is false if one only assumes that φ is continuous.

Hint: a) Prove it by contradiction: if $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ is such that $|x_n| \to \infty$ and $|\varphi(x_n)| \geq \delta > 0$ for every n, then the uniform continuity of φ implies the existence of $R > 0$ such that $|\varphi(x)| \geq \delta/2$ in $B(x_n, R)$. Show that this yields $\int_{\mathbb{R}^N} |\varphi|^p dx = \infty$. b) Consider the function $\varphi(x) = \sum_{n=1}^\infty f_n(x - n)$, where

$$f_n(x) = \begin{cases} nx + 1\,, & \text{if } -1/n \leq x \leq 0\,, \\ 1 - nx\,, & \text{if } 0 \leq x \leq 1/n\,, \\ 0\,, & \text{if } x \notin (-1/n, 1/n)\,. \end{cases}$$

Solution: a) Let us suppose that $\lim_{|x|\to\infty} \varphi(x) \neq 0$. Then, given $\delta > 0$, there exists a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^N$ such that $|x_n| \to \infty$ and $|\varphi(x_n)| \geq \delta$ for every $n \in \mathbb{N}$. As φ is uniformly continuous we have that there exists $R = R(\delta)$ such that for all $n \in \mathbb{N}$, if $|x - x_n| < R$ then $|\varphi(x) - \varphi(x_n)| < \delta/2$. But then, if $|x - x_n| < R$,

$$\big||\varphi(x)| - |\varphi(x_n)|\big| \leq |\varphi(x) - \varphi(x_n)| \leq \frac{\delta}{2} \quad \Rightarrow \quad |\varphi(x_n)| - |\varphi(x)| < \frac{\delta}{2}$$

and so

$$|\varphi(x)| > |\varphi(x_n)| - \frac{\delta}{2} \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}\,, \qquad \forall x \in B(x_n, R)\,,\ \forall n \in \mathbb{N}\,.$$

Therefore,

$$\int_{\mathbb{R}^N} |\varphi(x)|^p dx \geq \sum_{n=1}^\infty \int_{B(x_n,R)} |\varphi(x)|^p dx \geq \sum_{n=1}^\infty \big(\frac{\delta}{2}\big)^p m(B(x_n, R)) = \infty\,,$$

since all the balls $B(x_n, R)$ have the same Lebesgue measure. But this is a contradiction with the assumption that $\varphi \in L^p(\mathbb{R}^N)$.

b) Let us consider the function $\varphi(x)$ given in the hint. Note that $f_n(0) = 1$ and f_n is continuous, for all $n \in \mathbb{N}$. Since the supports of the functions $f_n(x-n)$ are $[n-\frac{1}{n}, n+\frac{1}{n}]$ for all $n \in \mathbb{N}$, they are disjoint. But then, for each $k \in \mathbb{N}$, we have that $\varphi(k) = f_k(0) = 1$ and so $\lim_{k\to\infty} \varphi(k) = 1 \neq 0$.

Problem 2.5.6 Suppose that $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots$ and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. What relation exists between f and g?

Solution: As $\|f_n - f\|_p \to 0$ as $n \to \infty$ we now that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \to f$ as $k \to \infty$ almost everywhere. Let $A := \{x \in X : \lim_{k\to\infty} f_{n_k}(x) \neq f(x)\}$ and $B := \{x \in X : \lim_{n\to\infty} f_n(x) \neq g(x)\}$. Then $A \cup B$ has zero μ-measure and, if $x \notin A \cup B$, then $\lim_{k\to\infty} f_{n_k} = g(x)$. Hence, $f(x) = g(x)$ for $x \in A \cup B$ and so, $f = g$ almost everywhere.

Problem 2.5.7 Suppose $\mu(X) = 1$, and suppose f and g are positive measurable functions on X such that $fg \geq 1$. Prove that

$$\int_X f\,d\mu \;\cdot\; \int_X g\,d\mu \geq 1\,.$$

Hint: Use Cauchy-Schwarz ineguality.
Solution: By Cauchy-Schwarz ineguality and, since $\sqrt{fg} \geq 1$, we get that

$$\Big(\int_X f\,d\mu\Big)\Big(\int_X g\,d\mu\Big) \geq \Big(\int_X \sqrt{fg}\,d\mu\Big)^2 \geq \Big(\int_X 1\,d\mu\Big)^2 = \mu(X)^2 = 1\,.$$

Problem 2.5.8 Suppose $\mu(X) = 1$ and $h : X \longrightarrow [0, \infty]$ is measurable. If $A := \int_X h\,d\mu$, prove that

$$\sqrt{1 + A^2} \leq \int_X \sqrt{1 + h^2}\,d\mu \leq 1 + A\,.$$

If μ is Lebesgue measure on $[0, 1]$ and h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general X) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Hint: The first inequality follows from Jensen's inequality. The second one follows from the inequality $\sqrt{1 + x^2} \leq 1 + x$ for $x \geq 0$.
Solution: The function $\varphi(x) := \sqrt{1 + x^2}$ is convex because $\varphi''(x) = (1 + x^2)^{-3/2} > 0$. Hence, by Jensen's inequality:

$$\sqrt{1 + A^2} = \varphi(A) = \varphi\Big(\int_X h\,d\mu\Big) \leq \int_X (\varphi \circ h)\,d\mu = \int_X \sqrt{1 + h^2}\,d\mu\,.$$

On the other hand, for $x \geq 0$: $1 + x^2 \leq (1 + x)^2 \implies \sqrt{1 + x^2} \leq 1 + x$. Therefore,

$$\int_X \sqrt{1 + h^2}\,d\mu \leq \int_X (1 + h)\,d\mu = \mu(X) + \int_X h\,d\mu = 1 + A\,.$$

If $X = [0, 1]$, μ is Lebesgue measure and $h = f'$ is continuous, then $A = \int_0^1 f'(x)\,dx = f(1) - f(0)$. Hence, the second inequality means that the length of the graph of f is $\leq$ than the length of the longer path from $(0, f(0))$ to $(1, f(1))$ which is $1 + (f(1) - f(0))$, the sum of the legs of the right triangle with vertices $(0, f(0))$, $(1, f(0))$ and $(1, f(1))$. The first inequality means that the shortest path is the straight line joining $(0, f(0))$ with $(1, f(1))$.
These facts suggest that the second inequality is an equality iff $h = 0$ a.e., that is to say, iff f is constant a.e., and the first one is an equality iff $h = A$ a.e.. Indeed, second inequality is

equality $\iff \sqrt{1+x^2} = 1+x$, for $x \geq 0 \iff x = 0$; first inequality is equality $\iff$ we have equality in Jensen's inequality $\iff \varphi(A) = (\varphi \circ h)(x)$ a.e. $\iff \sqrt{1+A^2} = \sqrt{1+(h(x))^2}$ a.e. $\iff h(x) = A$ a.e..

Problem 2.5.9 Let f be a complex function, $f \neq 0$. Let us define the function $\varphi(p) = \|f\|_p^p$ for $0 < p < \infty$ and let $E = \{p : \varphi(p) < \infty\} = \{p : f \in L^p(\mu)\}$. Prove that

a) If $r < p < s$ and $r, s \in E$, then $p \in E$.

b) $\log \varphi$ is convex in E.

c) Part a) implies that E is connected. Is E necessarily open? and closed? Can E be constituted by a single point? Can E be a any connected subset of $(0, \infty)$?

d) If $r < p < s$, then $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$.

Hints: a) $t^p \leq \max(t^r, t^s) \leq t^r + t^s$. b) If $p = \lambda r + (1-\lambda)s$ with $0 < \lambda < 1$, apply Hölder's inequality (with the conjugate exponents $\alpha = 1/\lambda$ and $\beta = 1/(1-\lambda)$) to bound $\varphi(p)$ in terms of $\varphi(r)$ and $\varphi(s)$. d) Apply part b).

Solution: a) As $t^p \leq \max(t^r, t^s) \leq t^r + t^s \implies |f|^p \leq |f|^r + |f|^s \implies \varphi(p) \leq \varphi(r) + \varphi(s) < \infty$. Hence, $p \in E$.

b) Let $p = \lambda r + (1-\lambda)s$ with $0 < \lambda < 1$. As $\alpha = 1/\lambda$ and $\beta = 1/(1-\lambda)$ are conjugate exponents, by Hölder's inequality we get

$$\begin{aligned}\int_X |f|^p d\mu = \int_X |f|^{\lambda r} |f|^{(1-\lambda)s} d\mu &\leq \Big(\int_X (|f|^{\lambda r})^{1/\lambda} d\mu \Big)^{\lambda} \Big(\int_X (|f|^{(1-\lambda)s})^{1/(1-\lambda)} d\mu \Big)^{1-\lambda} \\ &= \Big(\int_X |f|^r d\mu \Big)^{\lambda} \Big(\int_X |f|^s d\mu \Big)^{1-\lambda}.\end{aligned} \tag{2}$$

Hence, $\varphi(p) \leq \varphi(r)^{\lambda} \varphi(s)^{1-\lambda}$ and so $\log \varphi(p) \leq \lambda\, \varphi(r) + (1-\lambda) \log \varphi(s)$, i.e. $\log \varphi$ is convex.

c) E can be open, closed, unbounded and, even a single point, as the following examples show:
- $X = [0,1]$, $f_1(x) = 1/x^{1/a} \implies E = (0, a)$.
- $X = [1, \infty)$, $f_2(x) = 1/x^{1/b} \implies E = (b, \infty)$.
- $X = [0, 1/2]$, $f_3(x) = 1/(x \log^2 x)^{1/c} \implies E = (0, c]$.
- $X = [e, \infty)$, $f_4(x) = 1/(x \log^2 x)^{1/d} \implies E = [d, \infty)$.
- $X = (0, \infty)$, $f_5(x) = 1/(x(\log^2 x + 1))^{1/p} \implies E = \{p\}$.

d) If $\|f\|_r \leq \|f\|_s$ then, by (2), $\|f\|_p^p \leq \|f\|_r^{\lambda r} \|f\|_s^{(1-\lambda s} \leq \|f\|_s^{\lambda r} \|f\|_s^{(1-\lambda s} = \|f\|_s^{\lambda r + (1-\lambda)s} = \|f\|_s^p$.

Problem 2.5.10* Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e. $\mu(X) = 1$.

a) Prove that if φ is strictly convex: $\varphi(\lambda x + (1-\lambda)y) < \lambda \varphi(x) + (1-\lambda)\varphi(y)$ for $0 < \lambda < 1$, then equality holds in Jensen's inequality,

$$\varphi\Big(\int_X f\, d\mu \Big) \leq \int_X (\varphi \circ f)\, d\mu\,, \qquad \text{for } f \in L^1(\mu)\,,$$

if and only if f is constant almost everywhere.

b) If $0 < p < q \leq \infty$ prove that $\|f\|_p \leq \|f\|_q$.

c) Use part a) to prove that $\|f\|_p = \|f\|_q$ if and only if f is constant almost everywhere.

d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp\Big(\int_X \log|f|\,d\mu\Big)$$

if $\exp(-\infty)$ is defined to be 0.

Hints: a) If $f \neq 0$ a.e., then there exists $c \in \mathbb{R}$ such that $A = \{x : |f(x)| > c\}$ has $0 < \mu(A) < 1$. Take $\lambda = \mu(A)$, $x = \frac{1}{\lambda}\int_A f\,d\mu$, $y = \frac{1}{1-\lambda}\int_{A^c} f\,d\mu$ and apply Jensen's inequality. To bound $\varphi(x)$ and $\varphi(y)$ apply again Jensen's inequality. Finally, deduce that Jensen's inequality for this f is strict. b) Apply Jensen's inequality to the convex function $\varphi(x) = x^t$ with $t = q/p > 1$. c) $\varphi(x) = x^t$ is strictly convex. d) Apply Jensen's inequality with $\varphi(x) = -\log x$ and use that $\log x \leq x - 1$ for $x \in (0,\infty)$ and that $(t^p - 1)/t \to \log t$ as $p \to 0$. Use a convergence theorem.

Problem 2.5.11** Suppose $1 < p < \infty$, $f \in L^p((0,\infty), \mathcal{B}, m)$ and let us define

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad (0 < x < \infty).$$

a) Prove that the mapping $f \to F$ carries L^p into L^p and more concretely, prove Hardy's inequality:

$$\|F\|_p \leq \frac{p}{p-1}\,\|f\|_p\,.$$

b) Prove that equality holds in Hardy's inequality iff $f = 0$ almost everywhere.

c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Hints: a) Assume first that $f \geq 0$ and $f \in C_c((0,\infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p\int_0^\infty F^{p-1}(x)xF'(x)\,dx\,.$$

Note that $xF' = f - F$ and apply Hölder's inequality to $\int F^{p-1}f$. Then derive the general case. b) If equality holds for $f \geq 0$ deduce that we must have equality in

$$\int_0^\infty F^p(x)\,dx = q\int_0^\infty f(x)F^{p-1}\,dx \leq q\|f\|_p\Big(\int_0^\infty F^p(x)\,dx\Big)^{1/q}$$

and therefore that $\exists\, \alpha \geq 0$ such that $\alpha f^p = F^p$, and from this that f is constant a.e. c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. d) If $f \in L^1$ and $f \neq 0$ a.e., then $\exists\, x_0$ such that $\int_0^{x_0} f(t)\,dt > 0$.

Problem 2.5.12 Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \leq p < \infty$ and let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $L^p(\mu)$ such that $f_n \to f$ almost everywhere, as $n \to \infty$.

a) If, for some $M \geq 0$, $\|f_n\|_p \leq M$ for all $n \in \mathbb{N}$, then $f \in L^p(\mu)$ and

$$\|f\|_p \leq \liminf_{n\to\infty} \|f_n\|_p\,.$$

b) If, for some $F \in L^p(\mu)$, $|f_n(x)| \leq |F(x)|$ for all $n \in \mathbb{N}$ and almost every $x \in X$, then $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

c) Prove that b) is false for $p = \infty$.

Hints: a) Use Fatou's lemma. b) Use dominated convergence theorem. c) Consider the sequence $f_n = \chi_{(0,1/n)}$ in $(0,1)$.

Solution: a) By Fatou's lemma:

$$\int_X |f|^p d\mu = \int_X \lim_{n\to\infty} |f_n|^p \, d\mu \le \liminf_{n\to\infty} \int_X |f_n|^p \, d\mu \le M^p < \infty \,.$$

Therefore, $f \in L^p(\mu)$ and $\|f\|_p \le \liminf_{n\to\infty} \|f_n\|_p$.

b) By the dominated convergence theorem:

$$\int_X |f|^p d\mu = \int_X \lim_{n\to\infty} |f_n|^p \, d\mu = \lim_{n\to\infty} \int_X |f_n|^p \, d\mu \le \int_X |F|^p \, d\mu < \infty \,.$$

Therefore, $f \in L^p(\mu)$ and, as $|f_n(x) - f(x)| \le |f_n(x)| + |f(x)|$, then

$$|f_n(x) - f(x)|^p \le 2^{p-1}\big(|f_n(x)|^p + |f(x)|^p\big) \le 2^p |F(x)|^p \in L^1(\mu) \,,$$

and again, by the dominated convergence theorem,

$$\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = \int_X \lim_{n\to\infty} |f_n - f|^p \, d\mu = \int_X 0 \, d\mu = 0 \,.$$

c) Let $X = [0,1]$, $f_n = \chi_{(0,1/n)}$. Then $f_n(x) \to 0$ as $n \to \infty$ almost everywhere, $\|f_n\|_\infty = 1$, $\|f_n - f\|_\infty = \|f_n\|_\infty = 1$ and so $\|f_n - f\|_\infty$ does not converge to 0 as $n \to \infty$.

Problem 2.5.13* Let $0 < p < \infty$ and $f, f_n \in L^p(X, \mathcal{A}, \mu)$.

a) If $1 \leq p \leq \infty$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$, prove that $\|f_n\|_p \to \|f\|_p$.

b) Let $c_p = \max\{1, 2^{p-1}\}$. Prove that

$$|a - b|^p \leq c_p \left(|a|^p + |b|^p\right)$$

for arbitrary complex numbers a and b.

c) If $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$ prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

d) Prove that the conclusion of c) is false if the hypothesis $\|f_n\|_p \to \|f\|_p$ is removed, even if $\mu(X) < \infty$.

e) Prove that the conclusion of c) is false if $p = \infty$

Hint: a) Prove that $\big|\|f\|_p - \|g\|_p\big| \leq \|f - g\|_p$ for $f, g \in L^p(\mu)$. b) Prove the cases $0 < p \leq 1$ and $1 < p < \infty$ separately. For the first one, consider the function $\phi(x) = (x + y)^p - x^p - y^p$ for $x \geq 0$ and fixed $y \geq 0$ and prove that φ is decreasing. For the second case, consider the function $\psi(x) = 2^{p-1}(x^p + y^p) - (x + y)^p$ for $x \geq 0$ and fixed $y \geq 0$ and prove that ψ has an absolute minimum when $x = y$. c) Consider the function $h_n = c_p\left(|f|^p + |f_n|^p\right) - |f - f_n|^p$ and use Fatou's lemma as in the proof of the dominated convergence theorem.