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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.1: Continuity and differentiability

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3 Parametric integrals

3.1 Continuity and differentiability

Problem 3.1.1 Let $f(x,y) = \log(x^2 + y^2)$ for $y \in (0,1)$ and $x > 0$.

a) Prove that $\varphi(x) = \int_0^1 f(x,y)\,dy$ is well defined and is derivable. Prove that $\varphi'(x) = \int_0^1 \frac{\partial f}{\partial x}\,dy$ and calculate $\varphi'(x)$.

b) Prove that $\varphi(x)$ is continuous at $x_0 = 0$ and that $\varphi(0) = -2$.

c) Compute $\varphi(x)$ integrating by parts.

Hint: $f(x,\cdot)$ is continuous on $[0,1]$ for fixed $x > 0$. Besides $\left|\frac{\partial}{\partial x}\big[f(x,y)\big]\right| \le \frac{2}{x_0} \in L^1(0,1)$ for $x \ge x_0 > 0$. Hence, F is derivable on (x_0,∞) for all $x_0 > 0$ and so it is derivable on $(0,\infty)$.

Solution: a) For each fixed $x > 0$, the function $f_x(y) = \log(x^2 + y^2)$ is continuous on $[0,1]$. Hence, $\varphi(x)$ is well defined. Now, fixed $x_0 > 0$, we have

$$\frac{\partial}{\partial x}\big[\log(x^2+y^2)\big] = \frac{2x}{x^2+y^2} \le \frac{2x}{x^2} = \frac{2}{x} \le \frac{2}{x_0} \in L^1(0,1)\,, \qquad \text{if } x > x_0\,.$$

Hence, by the theorem on differentiation of parametric integrals, we have that $\varphi(x)$ is derivable on $(x_0,1)$, for all $x_0 > 0$. Therefore $\varphi(x)$ is derivable on $(0,1)$ and

$$\varphi'(x) = \int_0^1 \frac{\partial}{\partial x}\big[\log(x^2+y^2)\big]\,dy = \int_0^1 \frac{2x}{x^2+y^2}\,dy = \Big[2\arctan\frac{y}{x}\Big]_{y=0}^{y=1} = 2\arctan\frac{1}{x} = \pi - 2\arctan x.$$

b) As $\log y^2 \le \log(x^2+y^2) \le \log(1+y^2)$ if $x \in [0,1]$ and f_x is increasing on $[0,1]$, we have that $|\log(x^2+y^2)| \le \max\{\log(1+y^2), |\log y^2|\} = \max\{\log(1+y^2), 2\log(1/y)\}$ for all $x \in (0,1]$. Now the equation $1 + y^2 = 1/y^2$ has the unique solution $y_0 = \sqrt{(\sqrt{5}-1)/2}$ in $(0,1)$, and therefore

$$|\log(x^2+y^2)| \le g(y) := \begin{cases} 2\log(1/y)\,, & \text{if } y \le y_0\,,\\ \log(1+y^2)\,, & \text{if } y \ge y_0\,. \end{cases}$$

But $\log(1+y^2)$ is continuous on $[y_0,1]$ and so, $g \in L^1[y_0,1]$. Also, using the monotone convergence theorem and integrating by parts:

$$\int_0^{y_0} \log\frac{1}{y}\,dy = \lim_{N\to\infty} \int_{1/N}^{y_0} \log\frac{1}{y}\,dy = \lim_{N\to\infty}\Big[y\log\frac{1}{y}\Big]_{y=1/N}^{y=y_0} + 1 < \infty$$

since, by L'Hopital rule, $\lim_{N\to\infty} N\log N = 0$. Therefore, $g \in L^1(0,1]$ and by the theorem on continuity of parametric integrals, φ is continuous at $x_0 = 0$ and

$$\varphi(0) = \lim_{x\to 0^+} \varphi(x) = \int_0^1 \lim_{x\to 0^+} \log(x^2+y^2)\,dy = \int_0^1 \log y^2 dy\,.$$

Using now the monotone convergence theorem and integrating again by parts we get that:

$$\varphi(0) = 2\lim_{N\to\infty} \int_{1/N}^1 \log y\,dy = 2\lim_{N\to\infty}\big[y\log y\big]_{y=1/N}^{y=1} - 2 = 2\lim_{N\to\infty}\frac{1}{N}\log N - 2 = 0 - 2 = -2\,.$$

c) Integrating by parts taking $u = \arctan x,\ v' = 1 \implies u' = 1/(1+x^2),\ v = x$:

$$\begin{aligned}\varphi(x) &= \pi x - 2\int \arctan x\, dx = \pi x - 2x\arctan x + \int \frac{2x}{1+x^2}\, dx \\ &= \pi x - 2x\arctan x + \log(x^2+1) + c\,.\end{aligned}$$

But using that $\varphi(0) = -2$ we obtain that $c = -2$. Hence, $\varphi(x) = \pi x - 2x\arctan x + \log(x^2+1) - 2$.

Problem 3.1.2 Let $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$F(x) = \Big(\int_0^x e^{-t^2} dt\Big)^2 \qquad \text{and} \qquad G(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt\,.$$

Prove that:

a) $F'(x)+G'(x) = 0$, for all $x \in \mathbb{R}$. Justify why you can apply the theorem on differentiation of parametric integrals.

b) $F(x) + G(x) = \pi/4$, for all $x \in \mathbb{R}$.

c) Deduce that $\displaystyle\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Hints: a) $\big|\frac{\partial}{\partial x}\big[\frac{e^{-x^2(1+t^2)}}{1+t^2}\big]\big| = |2xe^{-x^2(1+t^2)}| \le 2 \in L^1[0,1]$ for $x \in \mathbb{R}$. c) Let $x \to \infty$ in b) by applying monotone convergence.

Solution: a) Using the Fundamental Theorem of Calculus we have that F is derivable on $\mathbb{R}$ and $F'(x) = 2e^{-x^2}\int_0^x e^{-t^2} dt$. On the other hand, for all $x \in \mathbb{R}$:

$$\Big|\frac{\partial}{\partial x}\Big[\frac{e^{-x^2(1+t^2)}}{1+t^2}\Big]\Big| = \big|2xe^{-x^2(1+t^2)}\big| \le 2 \in L^1[0,1]\,.$$

Hence, using the theorem on differentiation of parametric integrals, G is derivable on $\mathbb{R}$ and:

$$G'(x) = \int_0^1 \frac{\partial}{\partial x}\Big[\frac{e^{-x^2(1+t^2)}}{1+t^2}\Big]\, dt = -2x\int_0^1 e^{-x^2(1+t^2)}\, dt = -2xe^{-x^2}\int_0^1 e^{-x^2t^2}\, dt = -2e^{-x^2}\int_0^x e^{-t^2} dt\,,$$

where we have done the change of variable $u = xt$.

b) As $F'(x) + G'(x) = 0$ for all $x \in (0,\infty)$, we deduce that $F(x) + G(x) = k \in \mathbb{R}$. But then $k = F(0) + G(0) = 0 + [\arctan t]_{t=0}^{t=1} = \arctan 1 = \pi/4$.

c) We have that $\lim_{x\to\infty}\big(F(x) + G(x)\big) = \pi/4$ and so, by the monotone convergence theorem,

$$\Big(\int_0^\infty e^{-t^2} dt\Big)^2 + \lim_{x\to\infty}\int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt = \frac{\pi}{4} \implies \Big(\int_0^\infty e^{-t^2} dt\Big)^2 + 0 = \frac{\pi}{4}\,.$$

Problem 3.1.3 Calculate $F(s) = \displaystyle\int_0^\infty e^{-x}\sin(sx)\, dx$, and, justifying all the steps, from the obtained result calculate

$$G(s) = \int_0^\infty x\, e^{-x}\cos(sx)\, dx\,.$$

Hints: Use integration by parts to evaluate $F(s)$; $G(s)$ is derivable since $\left|\frac{\partial}{\partial s}\left[e^{-x}\sin(sx)\right]\right| \le x\,e^{-x} \in L^1(0,\infty)$.

Solution: First of all, integrating twice by parts and using the monotone convergence theorem, it is easy to obtain that $F(s) = s/(1+s^2)$.

On the other hand, as $\left|\frac{\partial}{\partial s}\left[e^{-x}\sin(sx)\right]\right| = |xe^{-x}\cos(sx)| \le x\,e^{-x} \in L^1(0,\infty)$, by the theorem on differentiation of parametric integrals we have that

$$F'(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-x}\sin(sx)\right]dx = \int_0^\infty xe^{-x}\cos(sx)\,dx = G(s) \implies G(s) = \frac{d}{ds}\left[\frac{s}{1+s^2}\right] = \frac{1-s^2}{(1+s^2)^2}\,.$$

Problem 3.1.4

a) Assuming that we can apply the Fundamental Theorem of Calculus and the theorem on parametric derivation, prove that:

$$F(x) = \int_a^{f(x)} g(x,t)\,dt \qquad \Longrightarrow \qquad F'(x) = g(x, f(x))\,f'(x) + \int_a^{f(x)} \frac{\partial g}{\partial x}(x,t)\,dt\,.$$

b) Prove that

$$\int_0^{\pi/(4a)} \frac{x}{\cos^2 ax}\,dx = \frac{1}{2a^2}\left(\frac{\pi}{2} - \log 2\right), \qquad \text{for } a > 0.$$

Hints: a) Consider the function $G(u,v) = \int_a^v g(u,t)\,dt$ and apply the chain rule. b) Use the previous part to calculate the derivative of $\int_0^{\pi/(4a)} \tan ax\,dx$ with respect to a.

Solution: a) Let $G(u,v) = \int_a^u g(v,t)\,dt$. Then, by the Fundamental Theorem of Calculus, $\frac{\partial G}{\partial u} = g(v,u)$ and, by the theorem on differentiation of parametric integrals, $\frac{\partial G}{\partial v} = \int_a^u \frac{\partial g}{\partial v}(v,t)\,dt$. Finally, as $F(x) = G(f(x), x)$, $u = f(x)$, $v = x$, by the chain rule:

$$F'(x) = \frac{\partial G}{\partial u}\,f'(x) + \frac{\partial G}{\partial v} = g(x, f(x))\,f'(x) + \int_a^{f(x)} \frac{\partial g}{\partial x}(x,t)\,dt\,.$$

b) Let $F(a) := \displaystyle\int_0^{\pi/(4a)} \tan ax\,dx$. First of all, let us observe that if $x \in [0, \pi/(4a)]$, then $ax \in [0, \pi/4]$ and $\tan ax$ is continuous on this interval. Hence, $F(a)$ is well-defined. Secondly, we can compute the value of $F(a)$:

$$F(a) = \int_0^{\pi/(4a)} \frac{\sin ax}{\cos ax}\,dx = -\frac{1}{a}\left[\log(\cos ax)\right]_{x=0}^{x=\pi/(4a)} = \frac{1}{2a}\,\log 2\,.$$

Thirdly, let us fix $a_0 > 0$ and let $a > a_0 > 0$. As

$$\left|\frac{\partial}{\partial a}(\tan ax)\right| = \frac{x}{\cos^2 ax} \le \frac{x}{\cos^2(\pi/4)} = 2x \in L^1[0, \pi/(4a_0)]\,,$$

using part a) we obtain that $F(a)$ is derivable on (a_0, ∞) for every $a_0 > 0$ and so, derivable on $(0,\infty)$ and

$$F'(a) = \tan\left(a\,\frac{\pi}{4a}\right)\left(\frac{-\pi}{4a^2}\right) + \int_0^{\frac{\pi}{4a}} \frac{x}{\cos^2 ax}\,dx \implies \int_0^{\frac{\pi}{4a}} \frac{x}{\cos^2 ax}\,dx = \frac{\pi}{4a^2} - \frac{1}{2a^2}\log 2 = \frac{1}{2a^2}\left(\frac{\pi}{2} - \log 2\right).$$

Problem 3.1.5 Prove that

$$J(a) = \int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{\pi+2}{8a^3}\,, \qquad \text{for } a>0.$$

Hint: $\left|\frac{\partial}{\partial a}\left[\frac{1}{x^2+a^2}\right]\right| = \frac{2a}{(x^2+a^2)^2} \le \frac{2M}{(x^2+\varepsilon^2)^2} \in L^1(0,\infty)$ for $a\in[\varepsilon,M]$.

Solution: Let $F(a) := \displaystyle\int_0^a \frac{dx}{x^2+a^2}$ for $a>0$.
First of all, $\left|\frac{\partial}{\partial a}\left[\frac{1}{x^2+a^2}\right]\right| = \frac{2a}{(x^2+a^2)^2} \le \frac{2M}{(x^2+\varepsilon^2)^2} \in L^1(0,\infty)$ for $a\in[\varepsilon,M]$, and so, by the theorem on differentiation of parametric integrals, F is derivable on (ε,M) for all $\varepsilon, M>0$. Hence, F is derivable on $(0,\infty)$ and by part a) of problem 3.1.4 we have that

$$F'(a) = \frac{1}{2a^2} - 2a\int_0^a \frac{dx}{(x^2+a^2)^2}\,.$$

But $F(a) = \frac{1}{a}\left[\arctan\frac{x}{a}\right]_{x=0}^{x=a} = \frac{\pi}{4a}$. Therefore

$$-\frac{\pi}{4a^2} = \frac{1}{2a^2} - 2a\int_0^a \frac{dx}{(x^2+a^2)^2} \implies \int_0^a \frac{dx}{(x^2+a^2)^2} = \frac{\pi+2}{8a^3}\,.$$

Problem 3.1.6 Let $F(\alpha) = \displaystyle\int_0^\infty \frac{e^{-\alpha x}-e^{-x}}{x}\,dx$.

a) Study when the integral converges.

b) Calculate $F'(\alpha)$ explicitly and then calculate $F(\alpha)$.

c) Obtain the successive derivatives $F^{(k)}(\alpha)$ and calculate $\displaystyle\int_0^\infty x^n e^{-x}dx$.

Hints: a) $\lim_{x\to 0^+}\frac{e^{-\alpha x}-e^{-x}}{x} = 1-\alpha$ and so, $\int_0^1 \frac{e^{-\alpha x}-e^{-x}}{x}\,dx < \infty$. Also, $\int_1^\infty \left|\frac{e^{-\alpha x}-e^{-x}}{x}\right| dx \le \int_0^\infty (e^{-\alpha x}+e^{-x})\,dx < \infty$ if $\alpha>0$. b) $\left|\frac{\partial}{\partial\alpha}\left[\frac{e^{-\alpha x}-e^{-x}}{x}\right]\right| \le e^{-\alpha_0 x} \in L^1(0,\infty)$ for $\alpha>\alpha_0>0$ and so F is derivable on (α_0,∞) for all $\alpha_0>0$. c) Derive both members of the identity $F'(\alpha) = -\int_0^\infty e^{-\alpha x}dx = -1/\alpha$.

Solution: a) Using L'Hopital rule we get that $\lim_{x\to0^+}(e^{-\alpha x}-e^{-x})/x = \lim_{x\to0^+} -\alpha e^{-\alpha x}+e^{-x} = 1-\alpha$. Hence, $f(x) = (e^{-\alpha x}-e^{-x})/x$ is continuous at $x=0$ (defining $f(0)=1-\alpha$) and so, $\int_0^1((e^{-\alpha x}-e^{-x})/x)\,dx<\infty$. Also, $\int_1^\infty \left|(e^{-\alpha x}-e^{-x})/x\right| dx \le \int_1^\infty (e^{-\alpha x}+e^{-x})\,dx < \infty$ if $\alpha>0$. Finally, if $\alpha<0$, then $\lim_{x\to+\infty} f(x) = \infty$ and, if $\alpha=0$ then, as $\lim_{x\to+\infty} e^{-x}=0$, we have that $(1-e^{-x})/x > (1-\varepsilon)/x$ for $x>M=M(\varepsilon)$ and so $\int_M^\infty[(1-e^{-x})/x]\,dx=\infty$. Therefore, $F(\alpha)$ converges (and so is well-defined) only for $\alpha>0$.
b) We have that $\left|\frac{\partial}{\partial\alpha}\left[\frac{e^{-\alpha x}-e^{-x}}{x}\right]\right| = e^{-\alpha x} \le e^{-\alpha_0 x} \in L^1(0,\infty)$ for $\alpha>\alpha_0>0$ and so F is derivable on (α_0,∞) for all $\alpha_0>0$. Hence, F is derivable on $(0,\infty)$ and, as in problem 2.1.8, we have that

$$F'(\alpha) = \int_0^\infty \frac{\partial}{\partial\alpha}\left[\frac{e^{-\alpha x}-e^{-x}}{x}\right] dx = -\int_0^\infty e^{-\alpha x}dx = -\frac{1}{\alpha}\,.$$

Hence, $F(\alpha) = -\int\frac{1}{\alpha}\,d\alpha = c-\log\alpha$. But, if $\alpha=1$, then $f(x)=0$ and so, $F(1)=0$. Hence, $F(\alpha) = -\log\alpha = \log\frac{1}{\alpha}$.

c) It is easy to prove by induction that $F^{(k)}(\alpha) = (-1)^k \frac{(k-1)!}{\alpha^k}$ for all $k \in \mathbb{N}$. On the other hand, we have that for $\alpha > \alpha_0$ and $k \in \mathbb{N}$:

$$\Big|\frac{\partial^k}{\partial a^k}(x^{k-1}e^{-\alpha x})\Big| = x^k e^{-\alpha x} \le x^k e^{-\alpha_0 x} \in L^1(0,\infty)\,.$$

Hence, by the theorem on differentiation of parametric integrals, $F^{(k)}(\alpha)$ is derivable on (α_0,∞) for all $\alpha_0 > 0$ and so it is derivable on $(0,\infty)$ and, for all $k \ge 1$,

$$F^{(k+1)}(\alpha) = (-1)^k \int_0^\infty \frac{\partial}{\partial a}(x^{k-1}e^{-\alpha x})\,dx = (-1)^{k+1}\int_0^\infty x^k e^{-\alpha x}\,dx \implies \int_0^\infty x^k e^{-\alpha x}\,dx = \frac{k!}{\alpha^{k+1}}\,.$$

Problem 3.1.7 Prove that for $a > 0$ and $b > 0$:

$$F(a,b) = \int_0^\infty (e^{-a^2/x^2} - e^{-b^2/x^2})\,dx = \sqrt{\pi}(b-a)\,.$$

Hint: $\big|\frac{\partial}{\partial a}\big[e^{-a^2/x^2} - e^{-b^2/x^2}\big]\big| \le \frac{2a}{x^2}e^{-a_0^2/x^2} \in L^1(0,\infty)$ for $a \ge a_0 > 0$. Hence, F is derivable on $[a_0,\infty)$ for all $a_0 > 0$ and so it is derivable on $(0,\infty)$. To compute $\frac{\partial}{\partial a}F(a,b)$ change variables to $t = 1/x$. Recall that $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}/2$ and observe that $F(a,a) = 0$.

Solution: First of all, $F(a,b)$ is well-defined since making the change of variable $t = 1/x$:

$$\int_0^1 e^{-a^2/x^2}\,dx = \int_1^\infty \frac{1}{t^2}\,e^{-a^2t^2}\,dt \le \int_1^\infty e^{-a^2t^2}\,dt \le \int_1^\infty e^{-a^2 t}\,dt < \infty$$

by problem 2.1.8, and similarly $\int_0^1 e^{-b^2/x^2}\,dx < \infty$. Besides, using again problem 2.1.8,

$$\int_1^\infty \left(e^{-a^2/x^2} - e^{-b^2/x^2}\right)dx \le \int_1^\infty \frac{C}{x^2}\,dx < \infty\,,$$

because, from the Taylor expansion of e^t, we have that $e^{-a^2/x^2} - e^{-b^2/x^2} = \frac{b^2-a^2}{x^2} + o(\frac{1}{x^2}) \le \frac{C}{x^2}$, $\forall\, x \in \mathbb{R}$.

On the other hand, we have that $\big|\frac{\partial}{\partial a}\big[e^{-a^2/x^2} - e^{-b^2/x^2}\big]\big| = \frac{2a}{x^2}e^{-a^2/x^2} \le \frac{2a}{x^2}e^{-a_0^2/x^2} \in L^1(0,\infty)$ for $a \ge a_0 > 0$, since making $t = 1/x$ we get that

$$\int_1^\infty \frac{1}{x^2}\,e^{-a_0^2/x^2}\,dx = \int_0^1 e^{-a_0^2t^2}\,dt < \infty\,,$$

because $e^{-a_0^2t^2}$ is continuous on $[0,1]$, and

$$\int_0^1 \frac{1}{x^2}\,e^{-a_0^2/x^2}\,dx = \int_1^\infty e^{-a_0^2t^2}\,dt \le \int_1^\infty e^{-a_0^2 t}\,dt < \infty\,,$$

by problem 2.1.8. Hence, by the theorem on differentiation of parametric integrals, F is derivable with respect to a on (a_0,∞) for all $a_0 > 0$ and so it is derivable on $(0,\infty)$. Besides,

$$\frac{\partial F}{\partial a} = -\int_0^\infty \frac{2a}{x^2}\,e^{-a^2/x^2}\,dx = -2\int_0^\infty e^{-t^2}\,dt = -\sqrt{\pi}\,,$$

where we have done the change of variable $t = a/x$ and we have used the problem 2.3.1. Hence, $F(a,b) = -\sqrt{\pi}\,a + C(b)$. But for $a = b$ it is clear that $F(a,a) = 0 \implies -\sqrt{\pi}\,b + C(b) = 0$ and so, $C(b) = \sqrt{\pi}\,b$ and finally, we obtain that $F(a,b) = \sqrt{\pi}\,(b-a)$.

Problem 3.1.8 Explain in the following cases why we can differentiate the parametric integral and why they are well-defined. Obtain explicitly the function deriving with respect to the parameter and integrating later with respect to it:

i) $F(s) = \displaystyle\int_0^{\pi/2} \log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{dx}{\cos x}$, with $|s|<1$.

ii) $G(a) = \displaystyle\int_0^{\infty} \log\left(1+\frac{a^2}{x^2}\right) dx$, with $a\in\mathbb{R}$.

iii) $H(p) = \displaystyle\int_0^{1} \frac{x^p-1}{\log x}\, dx$, with $p>-1$.

iv) $I(\lambda) = \displaystyle\int_0^{\pi/2} \frac{\log(1-\lambda^2\sin^2 x)}{\sin x}\, dx$, with $|\lambda|<1$.

v) $K(x) = \displaystyle\int_0^{\infty} e^{-t^2-x^2/t^2} dt$, with $x\in\mathbb{R}$.

Hints: i) $\left|\frac{\partial}{\partial s}\left[\log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{1}{\cos x}\right]\right| \le \frac{2}{1-s_0^2\cos^2 x} \in L^1(0,\pi/2)$ if $|s|\le s_0<1$. *ii)* Since G is an even function, it is enough to consider the case $a\ge 0$; $\left|\frac{\partial}{\partial a}\left[\log(1+\frac{a^2}{x^2})\right]\right| = \frac{2|a|}{x^2+a^2} \le \frac{2M}{x^2+\varepsilon^2} \in L^1(0,\infty)$ if $|a|\in[\varepsilon,M]$. *iii)* $\left|\frac{\partial}{\partial p}\left[\frac{x^p-1}{\log x}\right]\right| = x^p \in L^1(0,1)$ since $p>-1$. *iv)* $\left|\frac{\partial}{\partial\lambda}\left[\frac{\log(1-\lambda^2\sin^2 x)}{\sin x}\right]\right| = \frac{2|\lambda||\sin x|}{1-\lambda^2\sin^2 x} \le \frac{2}{1-\lambda_0\sin^2 x} \in L^1(0,\pi/2)$ if $|\lambda|<\lambda_0<1$. *v)* $\left|\frac{\partial}{\partial x}\left[e^{-t^2-x^2/t^2}\right]\right| \le \frac{2M}{t^2}\,(e^{-t^2}\chi_{[1,\infty)}(t) + e^{-\varepsilon^2/t^2}\chi_{(0,1)}(t)) \in L^1(0,\infty)$ if $|x|\in[\varepsilon,M]$. To compute $K'(x)$ change variables to $s=x/t$ and prove that $K'(x)=-2K(x)$. Note that $K(x)$ is even and so it is enough to compute it for $x\ge 0$.
Solutions: i) First of all, using L'Hopital rule,

$$\lim_{x\to\pi/2}\log\Big(\frac{1+s\cos x}{1-s\cos x}\Big)\frac{1}{\cos x} = \lim_{x\to\pi/2}\frac{\log(1+s\cos x)-\log(1-s\cos x)}{\cos x}$$
$$= \lim_{x\to\pi/2}\frac{\frac{-s\sin x}{1+s\cos x}-\frac{s\sin x}{1-s\cos x}}{-\sin x} = \lim_{x\to\pi/2}\Big(\frac{-s}{1+s\cos x}-\frac{s}{1-s\cos x}\Big) = -2s$$

and so, the integrand is continuous on $[0,\pi/2]$ and $F(s)$ is well defined. On the other hand, as

$$\Big|\frac{\partial}{\partial s}\Big[\log\Big(\frac{1+s\cos x}{1-s\cos x}\Big)\frac{1}{\cos x}\Big]\Big| = \Big|\Big[\frac{\cos x}{1+s\cos x}-\frac{-\cos x}{1-s\cos x}\Big]\frac{1}{\cos x}\Big|$$
$$= \frac{2}{1-s^2\cos^2 x} \le \frac{2}{1-s_0^2\cos^2 x} \in L^1(0,\pi/2)\,,$$

for $|s|\le s_0<1$. Hence, by the theorem on differentiation of parametric integrals, $F(s)$ is derivable on $(-s_0,s_0)$ for all $s_0<1$, and so $F(s)$ is derivable on $(-1,1)$. Besides,

$$F'(s) = \int_0^{\pi/2}\frac{\partial}{\partial s}\Big[\log\Big(\frac{1+s\cos x}{1-s\cos x}\Big)\frac{1}{\cos x}\Big]\,dx = \int_0^{\pi/2}\frac{2}{1-s^2\cos^2 x}\,dx$$

and making the change of variable $t=\tan x$, and using the monotone convergence theorem, we have that

$$F'(s) = \int_0^{\infty}\frac{2}{1-\frac{s^2}{1+t^2}}\,\frac{dt}{1+t^2} = \int_0^{\infty}\frac{2}{1-s^2+t^2}\,dt = \frac{2}{1-s^2}\int_0^{\infty}\frac{dt}{1+\big(\frac{t}{\sqrt{1-s^2}}\big)^2}$$
$$= \lim_{N\to\infty}\frac{2}{1-s^2}\int_0^{N}\frac{dt}{1+\big(\frac{t}{\sqrt{1-s^2}}\big)^2} = \lim_{N\to\infty}\frac{2}{\sqrt{1-s^2}}\Big[\arctan\Big(\frac{t}{\sqrt{1-s^2}}\Big)\Big]_{t=0}^{t=N} = \frac{\pi}{\sqrt{1-s^2}}\,.$$

Hence, $F(s) = \pi \arcsin s + c$. But, from the definition of $F(s)$ it is clear that $F(0) = 0$, and so $c = 0$. Therefore, $F(s) = \pi \arcsin s$.

ii) Using the Taylor expansion of $\log(1+t)$ around $t = 0$ we get that $\log\left(1 + \frac{a^2}{x^2}\right) = \frac{a^2}{x^2} + o\left(\frac{1}{x^2}\right)$ as $x \to \infty$ for each fixed $a \in \mathbb{R}$. Hence,

$$\int_1^\infty \log\left(1 + \frac{a^2}{x^2}\right) dx \le C \int_1^\infty \frac{1}{x^2}\, dx < \infty\,.$$

Also,

$$\int_0^1 \log\left(1 + \frac{a^2}{x^2}\right) dx = \int_0^1 \log(x^2 + a^2)\, dx - 2\int_0^1 \log x\, dx < \infty$$

since $\log(x^2+a^2)$ is continuous on $[0,1]$ if $a \neq 0$, and integrating by parts and using the monotone convergence theorem and L'Hopital rule:

$$\int_0^1 \log x\, dx = \lim_{\varepsilon \to 0^+} \left[x \log x - x\right]_{x=\varepsilon}^{x=1} = -1 - \lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = -1 > -\infty\,.$$

Hence, $\log(1 + a^2/x^2) \in L^1(0,\infty)$ and so, $G(a)$ is well-defined. On the other hand, as

$$\frac{\partial}{\partial a}\left[\log\left(1 + \frac{a^2}{x^2}\right)\right] = \frac{2a}{x^2 + a^2} \le \frac{2M}{x^2 + \varepsilon^2} \in L^1(0,\infty)$$

for all $a \in [\varepsilon, M]$ with $0 < \varepsilon < M < \infty$, using the theorem on differentiation of parametric integrals we deduce that $G(a)$ is derivable on (ε, M) for all ε and M and therefore, since G is also even, is derivable on $\mathbb{R} \setminus \{0\}$ and

$$G'(a) = \int_0^\infty \frac{\partial}{\partial a}\left[\log\left(1 + \frac{a^2}{x^2}\right)\right] dx = \int_0^\infty \frac{2a}{x^2 + a^2}\, dx\,.$$

Therefore,

$$G'(a) = \int_0^\infty \frac{2a}{x^2 + a^2}\, dx = 2\left[\arctan \frac{x}{a}\right]_{x=0}^{x=\infty} = 2\,\frac{\pi}{2} = \pi\,.$$

This implies that $G(a) = \pi a + c$ for $a > 0$, where c is a constant.

Since G is derivable on $\mathbb{R} \setminus \{0\}$, it is a continuous function on $\mathbb{R} \setminus \{0\}$. Let us prove that G is also continuous at 0: Consider a with $|a| < 1$. If $x \ge 1$, then

$$\log\left(1 + \frac{a^2}{x^2}\right) dx \le \frac{C}{x^2} \in L^1(1,\infty).$$

If $0 < x < 1$, then

$$\log\left(1 + \frac{a^2}{x^2}\right) dx \le \log\left(1 + \frac{1}{x^2}\right) dx \in L^1(0,1).$$

As G is continuous on $\mathbb{R}$, we deduce that $G(0) = c$. But, it is clear from the definition of G that $G(0) = 0$. Hence, $G(a) = \pi a$ for $a \ge 0$. Since $G(a)$ is an even function, we conclude that $G(a) = \pi |a|$ for $a \in \mathbb{R}$.

iii) First of all, if $p \ge 0$ we have that $\lim_{x \to 0^+} \frac{x^p - 1}{\log x} = 0$ and so the integrand is continuous on $[0,1]$. If $-1 < p < 0$, let $q = -p \in (0,1)$. Then

$$\frac{x^p - 1}{\log x} = \frac{x^{-q} - 1}{\log x} = \frac{1 - x^q}{\log x}\,\frac{1}{x^q} \le C\,\frac{1}{x^q} \in L^1[0,1]\,.$$

Hence, in any case, $H(p)$ is well defined for $p > -1$. On the other hand, as $p > -1$ we have that

$$\Big|\frac{\partial}{\partial p}\Big[\frac{x^p - 1}{\log x}\Big]\Big| = x^p \leq x^{p_0} \in L^1(0,1) \qquad \forall\, p \geq p_0 > -1$$

and so, by the theorem on derivation of parametric integrals, we conclude that $H(p)$ is derivable on (p_0, ∞) $\forall\, p_0 > -1$. Consequently, $H(p)$ is derivable on $(-1, \infty)$. Also, this same theorem gives that

$$H'(p) = \int_0^1 \frac{\partial}{\partial p}\Big[\frac{x^p - 1}{\log x}\Big]\,dx = \int_0^1 x^p\,dx = \Big[\frac{x^{p+1}}{p+1}\Big]_{x=0}^{x=1} = \frac{1}{p+1}\,,$$

and therefore

$$H(p) = \int \frac{1}{p+1}\,dp = \log(p+1) + c\,.$$

As $H(0) = \int_0^1 0\,dx = 0$, we obtain that $0 = H(0) = \log 1 + c = 0 + c = c$ and therefore $H(p) = \log(p+1)$.

iv) First of all, applying L'Hopital rule, we have that

$$\lim_{x\to 0^+} \frac{\log(1-\lambda^2\sin^2 x)}{\sin x} = \lim_{x\to 0^+} \frac{\frac{-2\lambda^2 \sin x\cos x}{1-\lambda^2\sin^2 x}}{\cos x} = 0$$

and so $\frac{\log(1-\lambda^2\sin^2 x)}{\sin x}$ is continuous on $[0,\pi/2]$ and $I(\lambda)$ is well-defined. On the other hand,

$$\Big|\frac{\partial}{\partial \lambda}\Big[\frac{\log(1-\lambda^2\sin^2 x)}{\sin x}\Big]\Big| = \frac{2|\lambda||\sin x|}{1-\lambda^2\sin^2 x} \leq \frac{2}{1-\lambda_0^2\sin^2 x} \in L^1(0,\pi/2)\,, \qquad \text{if } |\lambda| \leq \lambda_0 < 1$$

because $\frac{2}{1-\lambda_0^2\sin^2 x}$ is continuous on $[0,\pi/2]$. Hence, by the theorem on differentiability of parametric integrals, we have that $I(\lambda)$ is derivable on $(-\lambda_0, \lambda_0)$ for all $\lambda_0 \in (-1,1)$ and so it is derivable on $(-1,1)$. Besides,

$$I'(\lambda) = \int_0^{\pi/2} \frac{\partial}{\partial \lambda}\Big[\frac{\log(1-\lambda^2\sin^2 x)}{\sin x}\Big]\,dx = -\int_0^{\pi/2} \frac{2\lambda\sin x}{1-\lambda^2\sin^2 x}\,dx\,.$$

Changing variables to $t = \cos x$ we obtain that

$$\begin{aligned} I'(\lambda) &= -\int_0^1 \frac{2\lambda}{1-\lambda^2(1-t^2)}\,dt = -\int_0^1 \frac{2\lambda}{1-\lambda^2+\lambda^2 t^2}\,dt = -\frac{2\lambda}{1-\lambda^2}\int_0^1 \frac{dt}{1+\big(\frac{\lambda t}{\sqrt{1-\lambda^2}}\big)^2} \\ &= -\frac{2}{\sqrt{1-\lambda^2}}\Big[\arctan\frac{\lambda t}{\sqrt{1-\lambda^2}}\Big]_{t=0}^{t=1} = -\frac{2}{\sqrt{1-\lambda^2}}\arctan\frac{\lambda}{\sqrt{1-\lambda^2}}\,. \end{aligned}$$

But, if $\alpha := \arctan\frac{\lambda}{\sqrt{1-\lambda^2}} \implies \tan\alpha = \frac{\lambda}{\sqrt{1-\lambda^2}} \implies \sec^2\alpha = \frac{1}{1-\lambda^2} \implies \cos^2\alpha = 1-\lambda^2 \implies \alpha = \arcsin\lambda$ and so $I'(\lambda) = -\frac{2}{\sqrt{1-\lambda^2}}\arcsin\lambda \implies I(\lambda) = -(\arcsin\lambda)^2 + c$. But, from the definition of $I(\lambda)$, we have that $I(0) = 0$. Hence, $c = 0$ and so $I(\lambda) = -(\arcsin\lambda)^2$.

v) First of all $e^{-t^2-x^2/t^2} \leq e^{-t^2} \in L^1(0,\infty)$ and so $K(x)$ is well-defined and continuous on $\mathbb{R}$. Also, $K(x)$ is even and therefore, it is enough to compute it for $x \geq 0$. Now, given $0 < \varepsilon \leq x \leq M$, we have that

$$\Big|\frac{\partial}{\partial x}\big[e^{-t^2-x^2/t^2}\big]\Big| = \Big|-\frac{2x}{t^2}\,e^{-t^2-x^2/t^2}\Big| \leq \frac{2M}{t^2}\,\big(e^{-t^2}\chi_{[1,\infty)}(t) + e^{-\varepsilon^2/t^2}\chi_{(0,1)}(t)\big) \in L^1(0,\infty)\,,$$

Hence, by the theorem on differentiation of parametric integrals, $K(x)$ is derivable on (ε, M) for all $\varepsilon, M > 0$ and so it is derivable on $(0,\infty)$. Besides,

$$K'(x) = \int_0^\infty \frac{\partial}{\partial x}\big[e^{-t^2-x^2/t^2}\big]\,dt = -\int_0^\infty \frac{2x}{t^2}\,e^{-t^2-x^2/t^2}\,dt = -2\int_0^\infty e^{-x^2/s^2-s^2}\,ds = -2K(x)\,.$$

Hence, $K'(x)/K(x) = -2$ for $x > 0$ and so $\log K(x) = -2x + c \implies K(x) = C\,e^{-2x}$. But, from the definition, we have that $K(0) = \int_0^\infty e^{-t^2}\,dt = \sqrt{\pi}/2$ by problem 2.3.1. As K is continuous, we conclude that $C = \sqrt{\pi}/2$ and that $K(x) = \frac{\sqrt{\pi}}{2}\,e^{-2x}$ for $x \ge 0$ and, by symmetry, that $K(x) = \frac{\sqrt{\pi}}{2}\,e^{-2|x|}$ for $x \in \mathbb{R}$.

Problem 3.1.9 Obtain explicitly the function $F(t)$ justifying all the steps:

$$F(t) = \int_0^\infty e^{-tx}\,\frac{\sin x}{x}\,dx\,, \qquad \forall\, t > 0\,.$$

Hint: As $\big|\frac{\partial}{\partial t}\big[e^{-tx}\frac{\sin x}{x}\big]\big| \le e^{-tx} \le e^{-\varepsilon x} \in L^1(0,\infty)$ for $t \in (\varepsilon,\infty)$, we have that $F(t)$ is derivable on (ε,∞) for all $\varepsilon > 0$ and so it is derivable on $(0,\infty)$.

Solution: First of all, $\big|e^{-tx}\frac{\sin x}{x}\big| \le e^{-tx} \in L^1(0,\infty)$ because $|\sin x/x| \le 1$ and by problem 2.1.8. Hence, $F(t)$ is well defined. On the other hand,

$$\Big|\frac{\partial}{\partial t}\Big[e^{-tx}\,\frac{\sin x}{x}\Big]\Big| = \Big| -xe^{-tx}\,\frac{\sin x}{x}\Big| \le e^{-tx} \le e^{-\varepsilon x} \in L^1(0,\infty)$$

for all $t \in (\varepsilon,\infty)$. Hence, by the theorem on differentiation of parametric integrals, $F(t)$ is derivable on (ε,∞) for all $\varepsilon > 0$ and so it is derivable on $(0,\infty)$. Besides,

$$F'(t) = \int_0^\infty \frac{\partial}{\partial t}\Big[e^{-tx}\,\frac{\sin x}{x}\Big]\,dx = -\int_0^\infty e^{-tx}\sin x\,dx\,.$$

Integrating twice by parts and using the dominated convergence theorem we get that

$$\begin{aligned} F'(t) &= \lim_{N\to\infty}\big[e^{-tx}\cos x\big]_{x=0}^{x=N} + \lim_{N\to\infty} t\int_0^N e^{-tx}\cos x\,dx \\ &= -1 + \lim_{N\to\infty} t\big[e^{-tx}\sin x\big]_{x=0}^{x=N} + \lim_{N\to\infty} t^2\int_0^N e^{-tx}\sin x\,dx = -1 - t^2F'(t)\,. \end{aligned}$$

Hence, $F'(t) = -\frac{1}{1+t^2} \implies F(t) = c - \arctan t$. But, from the definition and using the dominated convergence theorem, it is easy to check that $\lim_{t\to\infty} F(t) = 0$. Therefore, as $\lim_{t\to\infty}\arctan t = \pi/2$, we conclude that $c = \pi/2$ and so that $F(t) = \frac{\pi}{2} - \arctan t$.

Problem 3.1.10 Prove that

$$\int_0^\infty \frac{1-\mathrm{e}^{-x^2}}{x^2}\,dx = \sqrt{\pi}.$$

Hint: Consider the function $F(t) = \int_0^\infty \frac{1-\mathrm{e}^{-tx^2}}{x^2}\,dx$ for $t > 0$ and proceed in a similar way to the previous problems.

Solution: Let $F(t) := \displaystyle\int_0^\infty \frac{1-\mathrm{e}^{-tx^2}}{x^2}\,dx$ for $t > 0$ and $f(x,t) = \dfrac{1-\mathrm{e}^{-tx^2}}{x^2}\,.$

First of all, as $\lim_{x\to 0^+} \frac{1-e^{-tx^2}}{x^2} = t$, we have that $f(x,t)$ is continuous on $x \in [0,1]$ and so $\int_0^1 \frac{1-\mathrm{e}^{-tx^2}}{x^2}\,dx < \infty$. Also $\int_1^\infty \frac{1-\mathrm{e}^{-tx^2}}{x^2}\,dx \le \int_1^\infty \frac{dx}{x^2} < \infty$. Hence, $F(t)$ is well-defined. On the other hand,

$$\frac{\partial}{\partial t}\Big[\frac{1-\mathrm{e}^{-tx^2}}{x^2}\Big] = e^{-tx^2} \le e^{-\varepsilon x^2} \in L^1(0,\infty)\,, \qquad \text{for all } t > \varepsilon\,,$$

since $e^{-\varepsilon x^2}$ is continuous and so integrable on $[0,1]$ and, if $x \in (1,\infty)$, then $e^{-\varepsilon x^2} \le e^{-\varepsilon x} \in L^1(1,\infty)$ by problem 2.1.8. Hence, by the theorem on differentiation of parametric integrals, $F(t)$ is derivable on (ε,∞) for all $\varepsilon > 0$, and so it is derivable on $(0,\infty)$. Besides,

$$F'(t) = \int_0^\infty \frac{\partial}{\partial t}\Big[\frac{1-\mathrm{e}^{-tx^2}}{x^2}\Big]\,dx = \int_0^\infty \mathrm{e}^{-tx^2}\,dx = \frac{1}{\sqrt{t}}\int_0^\infty e^{-u^2}\,du = \frac{1}{\sqrt{t}}\,\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{t}}\,,$$

where we have done the change of variable $u = \sqrt{t}\,x$ and we have used the problem 2.3.1. Hence, $F(t) = \sqrt{\pi t} + c$.
But, from the Taylor expansion of $f(x,t)$ around $x = 0$ we have, for given $\varepsilon > 0$, that there exists $\delta > 0$ such that $f(x,t) \le t + \varepsilon \le 1 + \varepsilon$ for $x \in (0,\delta)$ and $t \in [0,1]$. Hence, we have that

$$f(x,t) \le g(x) := (1+\varepsilon)\,\chi_{(0,\delta)} + \frac{1}{x^2}\,\chi_{[\delta,\infty)} \in L^1(0,\infty)$$

and so, by the theorem on continuity of parametric integrals, $F(t)$ is continuous on $[0,1]$. Besides,

$$0 = F(0) = \lim_{t\to 0^+} F(t) = \lim_{t\to 0^+} \sqrt{\pi t} + c = c \qquad \Longrightarrow \qquad c = 0$$

and

$$\int_0^\infty \frac{1-\mathrm{e}^{-x^2}}{x^2}\,dx = F(1) = \sqrt{\pi}\,.$$

Problem 3.1.11 Let $F(\lambda) = \displaystyle\int_0^\infty \frac{dx}{x^2+\lambda}$. Write the derivatives of F, and later prove that for all $\lambda > 0$,

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} = \frac{1\cdot 3\cdots(2n-1)}{2^n n!}\,\frac{\pi}{2\lambda^{n+1/2}} = \frac{(2n)!\,\pi}{(n!)^2(2\sqrt{\lambda}\,)^{2n+1}}\,.$$

Hints: First of all, it is easy to calculate $F(\lambda)$ and then all its derivatives $F^{(n)}(\lambda)$. Also, $\left|\frac{\partial}{\partial\lambda}\big[\frac{1}{x^2+\lambda}\big]\right| = \frac{1}{(x^2+\lambda)^2} \le \frac{1}{(x^2+\lambda_0)^2} \in L^1(0,\infty)$ for $\lambda > \lambda_0 > 0$. Hence, F is derivable on (λ_0,∞) for all $\lambda_0 > 0$ and so it is derivable on $(0,\infty)$. Similarly, we can see that F is infinitely derivable on $(0,\infty)$, and its derivatives can be calculated by parametric derivation: $F^{(n)}(\lambda) = \int_0^\infty \frac{\partial^n}{\partial\lambda^n}\big[\frac{1}{x^2+\lambda}\big]\,dx$.
Solution: First of all,

$$\Big|\frac{\partial}{\partial\lambda}\Big[\frac{1}{x^2+\lambda}\Big]\Big| = \frac{1}{(x^2+\lambda)^2} \le \frac{1}{(x^2+\lambda_0)^2} \in L^1(0,\infty)\,, \qquad \text{for all } \lambda > \lambda_0 > 0\,.$$

Hence, by the theorem on differentiation of parametric integrals, $F(\lambda)$ is derivable on (λ_0,∞) for all $\lambda_0 > 0$ and so it is derivable on $(0,\infty)$. Besides, $F'(\lambda) = \int_0^\infty \frac{-1}{(x^2+\lambda)^2}\,dx$. Similarly,

$$\Big|\frac{\partial}{\partial\lambda}\Big[\frac{1}{(x^2+\lambda)^2}\Big]\Big| = \frac{2}{(x^2+\lambda)^3} \le \frac{1}{(x^2+\lambda_0)^3} \in L^1(0,\infty)\,, \qquad \text{for all } \lambda > \lambda_0 > 0\,,$$

and $F'(\lambda)$ is again derivable on $(0,\infty)$ and $F''(\lambda) = \int_0^\infty \frac{2}{(x^2+\lambda)^3}$. Proceeding by induction, it is easy to obtain that $F(\lambda)$ is C^∞ on $(0,\infty)$ and, for all $n \in \mathbb{N}$,

$$F^{(n)}(\lambda) = (-1)^n n! \int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} \,. \tag{1}$$

But using the monotone convergence theorem and integrating directly, we have

$$F(\lambda) = \lim_{N\to\infty} \int_0^N \frac{dx}{x^2+\lambda} = \frac{1}{\sqrt{\lambda}} \lim_{N\to\infty} \Big[\arctan \frac{x}{\sqrt{\lambda}}\Big]_{x=0}^{x=N} = \frac{\pi}{2\sqrt{\lambda}}\,,$$

and proceeding again by induction, it is easy to obtain that, for all $n \in \mathbb{N}$,

$$F^{(n)}(\lambda) = (-1)^n \frac{\pi}{2} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(\sqrt{\lambda}\,)^{2n+1}} \,. \tag{2}$$

From (1) and (2) we obtain that

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} = \frac{(-1)^n}{n!} F^{(n)}(\lambda) = \frac{(2n)!\,\pi}{2^{n+1}2^n(n!)^2} \frac{1}{(\sqrt{\lambda}\,)^{2n+1}} = \frac{(2n)!\,\pi}{(n!)^2(2\sqrt{\lambda}\,)^{2n+1}} \,.$$

Problem 3.1.12 Let

$$F(x) = \int_0^{2x} \frac{\log(1+2xt)}{1+t^2}\, dt\,, \qquad x \geq 0\,.$$

a) Check that F is derivable on $(0,\infty)$ and prove that

$$F'(x) = \frac{\log(1+4x^2)}{1+4x^2} + \frac{4x}{1+4x^2} \arctan 2x\,.$$

b) Using the previous part, prove that

$$F(x) = \log\sqrt{1+4x^2}\, \arctan 2x\,.$$

Hints: a) $\big|\frac{\partial}{\partial x}\big[\frac{\log(1+2xt)}{1+t^2}\big]\big| \leq \frac{2t}{(1+t^2)(1+2x_0t)} \in L^1(0,\infty)$, for $x > x_0 > 0$. Hence, F is derivable on (x_0,∞) for all $x_0 > 0$ and so it is derivable on $(0,\infty)$. To calculate $F'(x)$ use decomposition on simple fractions. b) Integrate by parts.

Solution: a) First of all,

$$\frac{\partial}{\partial x}\Big[\frac{\log(1+2xt)}{1+t^2}\Big] = \frac{1}{1+t^2}\frac{2t}{1+2xt} \leq \frac{2t}{(1+t^2)(1+2x_0t)} \in L^1(0,\infty)\,, \qquad \text{for all } x \geq x_0 > 0\,.$$

Hence, by the theorem on differentiation of parametric integrals, $F(x)$ is derivable on (x_0,∞) for all $x_0 > 0$ and so it is derivable on $(0,\infty)$. Besides, by part a) of problem 3.1.4

$$F'(x) = 2\,\frac{\log(1+4x^2)}{1+4x^2} + \int_0^{2x} \frac{2t}{(1+t^2)(1+2xt)}\, dt\,.$$

Decomposing into simple fractions, we have that

$$\frac{2t}{(1+t^2)(1+2xt)} = \frac{At+B}{1+t^2} + \frac{C}{1+2xt}\,, \qquad \text{with } A = \frac{2}{1+4x^2}\,,\ B = -C = \frac{4x}{1+4x^2}\,.$$

Hence,

$$F'(x) = 2\,\frac{\log(1+4x^2)}{1+4x^2} + \frac{1}{1+4x^2}\int_0^{2x}\frac{2t}{1+t^2}\,dt + \frac{4x}{1+4x^2}\int_0^{2x}\frac{dt}{1+t^2} - \frac{2}{1+4x^2}\int_0^{2x}\frac{2x\,dt}{1+2xt}$$
$$= 2\,\frac{\log(1+4x^2)}{1+4x^2} + \frac{1}{1+4x^2}\big[\log(1+t^2)\big]_{t=0}^{t=2x} + \frac{4x}{1+4x^2}\big[\arctan t\big]_{t=0}^{t=2x}$$
$$- \frac{2}{1+4x^2}\big[\log(1+2xt)\big]_{t=0}^{t=2x} = \frac{\log(1+4x^2)}{1+4x^2} + \frac{4x}{1+4x^2}\ \arctan 2x\,.$$

b) Integrating by parts with $u=\log(1+4x^2)$ and $v'=1/(1+4x^2)$, we obtain that

$$\int\frac{\log(1+4x^2)}{1+4x^2}\,dx = \frac{1}{2}\,\log(1+4x^2)\arctan 2x - \int\frac{4x}{1+4x^2}\ \arctan 2x\,dx$$

and so

$$F(x) = \int\frac{\log(1+4x^2)}{1+4x^2}\,dx + \int\frac{4x}{1+4x^2}\ \arctan 2x\,dx = \log\sqrt{1+4x^2}\arctan 2x + c\,.$$

But, from the definition of $F(x)$, we know that $F(0)=0$ and so, $c=0$.

Problem 3.1.13* Prove that

$$\int_0^{\pi}\frac{\log(1+\cos x)}{\cos x}\,dx = \frac{\pi^2}{2},$$

calculating first

$$F(t) := \int_0^{\pi}\frac{\log(1+t\cos x)}{\cos x}\,dx \qquad \text{for } |t|\le 1\,.$$

Hints: $\left|\frac{\partial}{\partial t}\left[\frac{\log(1+t\cos x)}{\cos x}\right]\right| = \frac{1}{1+t\cos x}$ which is continuous for $|t|<1$, and so it belongs to $L^1(0,\pi)$. This means that $F(t)$ is derivable on $(-1,1)$. Compute $F(t)$ by using parametric derivation and calculate $F'(t)=\pi/\sqrt{1-t^2}$ (change variables to $u=\tan(x/2)$). Now, if $0\le t\le 1$, we have that $f(x,t)=\frac{\log(1+t\cos x)}{\cos x}$ verifies, for $x\in[0,\pi/2)$, that $f(x,t)\le\frac{\log(1+\cos x)}{\cos x}$ which is continuous at $x=\pi/2$ and so it belongs to $L^1[0,\pi/2)$, and for $x\in(\pi/2,\pi)$ that $f(x,t)\le g(x):=\frac{1}{|\cos x|}\log\frac{1}{1-|\cos x|}$. But $g(x)$ is continuous at $x=\pi/2$ and $\log\frac{1}{1-|\cos x|}\in L^1[\pi/2,\pi)$ since $\lim_{x\to\pi^-}\frac{\log(1+\cos x)}{(\pi-x)^{-\varepsilon}}=0$ for each $\varepsilon>0$. Hence, $F(t)$ is continuous on $[0,1]$ and $F(1)=\lim_{t\to 1^-}F(t)$.
Solution: $F(t)=\pi\,\mathrm{arcsen}\,t$.

Problem 3.1.14* Let us consider the function

$$F(x) = \int_0^1\frac{(\log(1-xt))^2}{t}\,dt.$$

a) Find the values of x such that $F(x)$ is defined.

b) Calculate $F'(x)$ justifying why you can derive. Evaluate the resulting integral.

c) Study the increasing and decreasing intervals of F.

Hints: a) As $\lim_{z\to 0^+}(\log(1-z))/z = 1$ we have $\log(1-z) \le Cz$ for $0 < z < \delta$. As $\lim_{z\to 0^+} z^\varepsilon \log z = 0$ we have $|\log z| \le z^{-\varepsilon}$ for $0 < z < \delta'$. b) If $x < x_0 < 1$, then $\frac{\partial}{\partial x}\big(\frac{(\log(1-xt))^2}{t}\big) \le 2\frac{1}{1-x_0 t}\log\frac{1}{1-x_0 t}$ which is continuous for $t \in [0,1]$. To evaluate F', integrate by parts.
Solution: a) $F(x) < \infty$ for $x \in (-\infty, 1]$. b) F is derivable for $x \in (-\infty, 1)$ and $F'(x) = (\log(1-x))^2/x$. c) F decreases on $(-\infty, 0)$ and increases on $(0,1)$.

Problem 3.1.15** Given $a > 0$, $b > 0$, prove that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\, dx = \frac{\pi}{2}(b-a)\,.$$

Hints: Consider the function $f(x,t) = \frac{\cos ax - \cos bx}{x^2}\, e^{-tx}$. Then $\big|\frac{\partial}{\partial t} f(x,t)\big| \le \frac{|\cos ax - \cos bx|}{x} e^{-t_0 x} \in L^1(0,\infty)$ for $t \ge t_0 > 0$. Hence, $F(t) = \int_0^\infty f(x,t)\, dx$ is derivable on $(0,\infty)$. Even more, as $\big|\frac{\partial^2}{\partial t^2} f(x,t)\big| \le 2e^{-t_0 x} \in L^1(0,\infty)$ for $t \ge t_0 > 0$, we also have that $F(t)$ is twice derivable on $(0,\infty)$. Also, as $|f(x,t)| \le \frac{|\cos ax - \cos bx|}{x^2} \in L^1(0,\infty)$ for $t \ge 0$, we have that F is continuous on $[0,\infty)$ and so, $F(0) = \lim_{t\to 0^+} F(t)$. To compute $F''(t)$, integrate by parts and prove that $F''(t) = \frac{t}{t^2+a^2} - \frac{t}{t^2+b^2}$. Hence, $F'(t) = \log\sqrt{\frac{t^2+a^2}{t^2+b^2}} + c_1$. By dominated convergence we have that $\lim_{t\to\infty} F'(t) = 0$ and so we deduce that $c_1 = 0$. Integrate again by parts to obtain $F(t) = t\log\sqrt{\frac{t^2+a^2}{t^2+b^2}} + a\arctan\frac{t}{a} - b\arctan\frac{t}{b} + c_2$. Finally, again by dominated convergence $\lim_{t\to\infty} F(t) = 0$ and so $c_2 = \frac{\pi}{2}(b-a)$, since $\lim_{t\to\infty} t\log\frac{t^2+a^2}{t^2+b^2} = 0$ by L'Hopital Rule.