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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 1: Measure theory

Section 1.3: Construction of measures

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1 Measure Theory

1.3. Construction of measures

Problem 1.3.1 Let μ^* be an outer measure in X and let H be a μ^*-measurable set. Let us consider the restriction μ_0^* of μ^* to $\mathcal{P}(H)$: $\mu_0^*(A) = \mu^*(A \cap H)$ for all $A \subset X$.

i) Check that μ_0^* is an outer measure on H.

ii) Check that $M \subseteq H$ is μ_0^*-measurable if and only if it is μ^*-measurable.

Problem 1.3.2

i) Let X be any set. Let us define $\mu^* : \mathcal{P}(X) :\longrightarrow [0,1]$ by $\mu^*(\varnothing) = 0$, $\mu^*(A) = 1$, if $A \neq \varnothing$, $A \subseteq X$. Check that μ^* is an outer measure and determine the σ-algebra $\mathcal{M}$ of measurable sets.

ii) Do the same if $\mu^*(\varnothing) = 0$, $\mu^*(A) = 1$, if $A \neq \varnothing$, $A \subsetneq X$, $\mu^*(X) = 2$.

Hints: i) If $\varnothing \subsetneq M \subsetneq X$, then the definition of μ^*-measurable set fails with $E = X$. ii) If $\operatorname{card}(X) > 2$ and $\{x, y\} \subset M \subsetneq X$ the definition fails with $E = M^c \cup \{x\}$; if $M = \{x\}$ the definition fails with $E = \{x, y\} \subsetneq X$.

Problem 1.3.3 Show that a finitely additive outer measure is countably additive.

Hint: $\bigcup_{j=1}^n A_j \subseteq \bigcup_{j=1}^\infty A_j$ for all n.

Problem 1.3.4* Let μ^* be an outer measure on X and let $\mathcal{M}$ be the collection of μ^*-measurable sets. Prove Caratheodory's theorem following the steps:

a) If $\mu^*(M) = 0$ then $M \in \mathcal{M}$.

b) If $M \in \mathcal{M}$ then also $M^c = X \setminus M \in \mathcal{M}$.

c) If $M, N \in \mathcal{M}$ then $M \cup N$, $M \cap N$, $M \setminus N \in \mathcal{M}$.

d) If $\{M_j\}_{j=1}^\infty$ is a sequence of disjoints in $\mathcal{M}$, then prove by induction on n that

$$\mu^*(A \cap (\cup_1^n M_j)) = \sum_{j=1}^n \mu^*(A \cap M_j)\,, \qquad \forall A \subset X\,, \forall n \in \mathbb{N}\,.$$

e) If $\{M_j\}_{j=1}^\infty$ is a sequence of disjoints in $\mathcal{M}$ and $M := \cup_{n=1}^\infty M_j$ then

$$\mu^*(A \cap M) = \sum_{j=1}^\infty \mu^*(A \cap M_j)\,, \qquad \forall A \subset X\,.$$

f) If $\{M_j\}_{j=1}^\infty$ is a sequence of disjoints in $\mathcal{M}$, them $M := \cup_{n=1}^\infty M_j \in \mathcal{M}$.

g) $\mathcal{M}$ is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a measure.

h) $(X, \mathcal{M}, \mu^*)$ is a complete measure space.

Hints: c) $A\cap(M\cup N) = (A\cap M)\cup(A\cap M^c\cap N)$. d) By c) $\cup_{j=1}^n M_j \in \mathcal{M}$ and so $\mu^*(A\cap(\cup_{j=1}^{n+1} M_j)) = \mu^*(A\cap(\cup_{j=1}^{n+1} M_j)\cap(\cup_{j=1}^{n} M_j)) + \mu^*(A\cap(\cup_{j=1}^{n+1} M_j)\setminus(\cup_{j=1}^{n} M_j)) = \mu^*(A\cap(\cup_{j=1}^{n} M_j)) + \mu^*(A\cap M_{n+1})$. e) It is a consequence of a). f) Use that $\cup_{j=1}^n M_j \in \mathcal{M}$ by c), and so $\mu^*(A) = \mu^*(A\cap\cup(\cup_{j=1}^n M_j)) + \mu^*(A\setminus(\cup_{j=1}^n M_j))$; use now parts d) and e). g) If $\{A_j\}$ is any collection of subsets in $\mathcal{M}$, then the sets $M_j = A_j \setminus (A_1 \cup \cdots \cup A_{n-1} \in \mathcal{M}$ are disjoints and $\cup_{j=1}^\infty A_j = \cup_{j=1}^\infty M_j$.

Problem 1.3.5* Let $\mathcal{E} \subset \mathcal{P}(X)$ be a semialgebra and let $\mu_0 : \mathcal{E} :\longrightarrow [0,\infty]$ be a countable additive set function.

a) Prove that μ_0 is monotone: If $E, F \in \mathcal{E}$, $E \subseteq F$, then $\mu_0(E) \le \mu_0(F)$.

b) Prove that μ_0 is countably sub-additive: If $E = \cup_{i=1}^\infty E_i$ with $E_i, E \in \mathcal{E}$, then

$$\mu_0(E) \le \sum_{i=1}^\infty \mu_0(E_i)\,.$$

c) Let us define, for all $A \subseteq X$,

$$\mu^*(A) = \inf\left\{ \sum_{i=1}^\infty \mu_0(E_i) :\ E_i \in \mathcal{E}\,,\ A \subseteq \cup_{i=1}^\infty E_i \right\}.$$

d) Prove that μ^* is an outer measure (and so, by Caratheodory's Theorem, the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra and $\mu = \mu^*|_{\mathcal{A}}$ is a complete measure).

e) Prove that $\mathcal{E} \subseteq \mathcal{A}$ and that μ^* is an extension of μ_0: $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{E}$.

Hints: a) If $E_1 \subset E_2$ then, as $\mathcal{E}$ is semialgebra, $E_2 = E_1 \cup E_1^c = E_1 \cup F_1 \cup \cdots F_n$ with $F_j \in \mathcal{E}$ and disjoint. b) Consider the disjoint sets $D_i := E_i \setminus (E_1 \cup \cdots \cup E_{i-1}) = E_i \cap (\cap_{i=1}^{n-1} E_i^c)$ and observe that, as $\mathcal{E}$ is semialgebra, we have that $E_i^c = F_{i1} \cup \cdots \cup F_{ik(i)}$ with $F_{ij} \in \mathcal{E}$ and disjoint. c) Given $\varepsilon > 0$ and sets $\{A_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \mu^*(A_i) < \infty$, choose for each i a collection $\{E_{ij}\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty \mu_0(E_{ij}) < \mu^*(A_i) + \varepsilon/2^i$. Then $A := \cup_i A_i \subseteq \cup_i \cup_j E_{ij}$ and $\mu^*(A) \le \sum_i \mu^*(A_i) + \varepsilon$. e) Given $E \in \mathcal{E}$, $A \subset X$ with $\mu^*(A) < \infty$ and $\varepsilon > 0$ there exists $\{E_i\} \subset \mathcal{E}$ such that $A \subset \cup_i E_i$ and $\sum_i \mu_0(E_i) < \mu^*(A) + \varepsilon$; also $E^c = F_1 \cup \cdots \cup F_n$ with $F_j \in \mathcal{E}$ and disjoint. Hence, $E_i = (E_i \cap E) \cup (E_i \cap F_1) \cup \cdots \cup (E_i \cap F_n)$, a disjoint union of sets.

Problem 1.3.6 A semiopen interval in $\mathbb{R}$ is an interval of type $\varnothing$, $[a,b)$, $(-\infty,b)$, $[a,\infty)$ or $(-\infty,\infty) = \mathbb{R}$. A semiopen interval in $\mathbb{R}^n$ is a set of type $I = I_1 \times I_2 \times \cdots \times I_n$, where each I_j is a semiopen interval in $\mathbb{R}$. Let $\mathcal{E}$ be the collection of semiopen intervals in $\mathbb{R}^n$. Prove that $\mathcal{E}$ is a semialgebra.

Problem 1.3.7 Show that a subset $B \subseteq \mathbb{R}$ is Lebesgue-measurable if and only if

$$m^*(I) = m^*(I \cap B) + m^*(I \cap B^c)\,,$$

for every open interval $I \subseteq \mathbb{R}$.

Hint: Given $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\varepsilon > 0$, consider a sequence of intervals $\{I_n\}$ such that $E \subset \cup_n I_n$ and $\sum_n m(I_n) < m^*(E) + \varepsilon$ and observe that, as each I_n is Lebesgue-measurable, $m(I_n) = m^*(I_n) = m^*(B \cap I_n) + m^*(B^c \cap I_n)$.

Problem 1.3.8*

a) Prove that $(\mathbb{R}^n, \mathcal{M}, m)$ is translations invariant:

$$A \in \mathcal{M}\,, \quad a \in \mathbb{R}^n \qquad \Longrightarrow \qquad a + A \in \mathcal{M} \quad \text{and} \quad m(a+A) = m(A)\,.$$

b) Let $(\mathbb{R}^n, \mathcal{M}, \mu)$ be a translations invariant measure space with μ a Radon measure ($\mu(K) < \infty$ for each compact set K). Prove that there exists $k \geq 0$ such that $\mu = km$.

Hints: a) Consider the measure $\mu(B) = m(a+B)$ for $B \in \mathcal{B}(\mathbb{R}^n)$ and observe that $m(a+I) = m(I)$ for each semi-interval I. Hence $\mu(I) = m(I)$ for I semi-interval. Apply Caratheodory-Hopf's extension theorem. b) Let $k = \mu([0,1] \times \cdots \times [0,1])$ and prove that $\mu(I) = k\, m(I)$, for each semi-interval $I = [0, r_1/q_1] \times \cdots \times [0, r_n/q_n]$ with $r_i/q_i \in \mathbb{Q}$. Using now an approximation argument conclude that the same is true for any semi-interval in $\mathbb{R}^n$. Finally apply Caratheodory-Hopf's extension theorem.

Problem 1.3.9* Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry for the Euclidean norm. that is to say $\|g(x) - g(y)\| = \|x - y\|$ for all $x, y \in \mathbb{R}^n$. It is known that any isometry is a composition of a translation and an orthogonal transformation. Recall that $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if U is linear and $UU^T = I$ where I is the identity matrix.
Prove that given any Lebesgue-measurable set M, then $g(M)$ is also a Lebesgue-measurable set and $m(g(M)) = m(M)$.

Hints: By problem 1 it suffices to prove it for an orthogonal transformation U. As U is an homeomorphism (bijective and continuous with continuous inverse) then U sends Borel sets into Borel sets. Define a measure μ by $\mu(A) = m(U(A))$ for $A \in \mathcal{B}(\mathbb{R}^n)$, where U is orthogonal, and prove that μ is translations invariant. Hence $\mu(A) = k\, m(A)$ for any $A \in \mathcal{B}(\mathbb{R}^n)$ and for some constant k. But, if $B = \{x : \|x\| < 1\}$ then prove that $\mu(B) = m(B)$ and so $k = 1$. Finally, if $M \in \mathcal{M}$ then $M = A \cup N$ with $A \in \mathcal{B}(\mathbb{R}^n)$ and $N \subset C \in \mathcal{B}(\mathbb{R}^n)$, $m(C) = 0$. Hence $U(M) = U(A) \cup U(N)$ with $U(A) \in \mathcal{B}(\mathbb{R}^n)$ and $U(N) \subset U(C) \in \mathcal{B}(\mathbb{R}^n)$, $m(U(C)) = m(C) = 0$.

Problem 1.3.10* Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation. Prove that given any Lebesgue-measurable set, then $T(M)$ is also a Lebesgue-measurable set and

$$m(T(M)) = |\det T|\, m(M).$$

Hints: If $\det T = 0$ is trivial because in this case $T(\mathbb{R}^n)$ is contained in an $(n-1)$-dimensional hyperplane which has zero n-dimensional Lebesgue measure. If $\det T \neq 0$, then T is bijective and can be decomposed as $T = UDV$ with U, V orthogonal transformations and D a linear transformation whose matrix is diagonal. As orthogonal transformations are isometries, by problem **??**, it suffices to prove it for D. Let $\lambda_1, \ldots, \lambda_n$ be the elements of the diagonal of D. If $I = [a_1, b_1) \times \cdots \times [a_n, b_n)$ is a semi-interval in $\mathbb{R}^n$, then $D(I) = [\lambda_1 a_1, \lambda_1 b_1) \times \cdots \times [\lambda_n a_n, \lambda_n b_n)$ and so $m(D(I)) = \lambda_1 \cdots \lambda_n\, m(I)$. Define the measure $\mu(M) = \frac{1}{\lambda_1 \cdots \lambda_n}\, m(D(M))$. By Caratheodory-Hopf's extension theorem we have that $\mu = m$. Finally, observe that $\det T = \det D = \lambda_1 \cdots \lambda_n$.

Problem 1.3.11* Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Prove that there exists a unique Radon measure $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0, \infty]$ such that

$$\mu([a,b)) = g(b^-) - g(a^-)\,, \qquad \forall\, [a,b) \in \mathcal{E}\,.$$

Here $g(x_0^-)$ denotes the left limit of g at the point x_0. This measure $\mu = \mu_g$ is called the *Borel-Stieltjes measure with distribution function g.*

Hint: Prove that μ is countably additive on the semi-intervals: if $[a,b) = \cup_{j=1}^{\infty}[a_j, b_j)$ then $g(b^-) - g(a^-) = \sum_{j=1}^{\infty} g(b_j^-) - g(a_j^-)$. Then, apply Caratheodory-Hopf's extension theorem.

Problem 1.3.12 Let $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$ be a Radon measure. Prove that there exists an increasing and left-continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mu = \mu_g$. Besides, g is unique unless by adding constants.

Hint: Define $g(t) = \mu([0,t])$ for $t \geq 0$ and $g(t) = -\mu([t,0))$ for $t < 0$ and apply Caratheodory-Hopf's extension theorem.

Problem 1.3.13 Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function and let μ_g be the corresponding Borel-Stieltjes measure with distribution function g. Prove that:

a) $\mu_g(\{x\}) = g(x^+) - g(x^-)$.

b) $\mu_g(\{x\}) = 0$ if and only if g is continuous at x.

c) $\mu_g([a,b]) = g(b^+) - g(a^-)$.

d) $\mu_g((a,b)) = g(b^-) - g(a^+)$.

e) $\mu_g((a,b]) = g(b^+) - g(a^+)$.

f) If $I \subset \mathbb{R}$ is an open interval, then $\mu_g(I) = 0$ if and only if g is constant on I.

Problem 1.3.14

a) Let us consider the function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ x & \text{if} \quad 1 \leq x < 3 \\ 4 & \text{if} \quad x \geq 3\,. \end{cases}$$

Let μ_F be the Borel-Stieltjes measure with distribution function F. Calculate:

$$\mu_F(\{1\}), \quad \mu_F(\{2\}), \quad \mu_F(\{3\}), \quad \mu_F((1,3]), \quad \mu_F((1,3)), \quad \mu_F([1,3]), \quad \mu_F([1,3)).$$

b) Give an example of a distribution function F such that

$$\mu_F((a,b)) < F(b) - F(a) < \mu_F([a,b])\,, \qquad \text{for some } a \text{ and } b.$$

Problem 1.3.15 Let $F(x)$ be the distribution function on $\mathbb{R}$ given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty,-1) \\ 1+x & \text{if} \quad x \in [-1,0) \\ 2+x^2 & \text{if} \quad x \in [0,2) \\ 9 & \text{if} \quad x \in [2,\infty)\,. \end{cases}$$

If μ_F is the Borel-Stieltjes measure with distribution function F, calculate the measure μ_F of the following sets: $\{2\}$, $[-1/2,3)$, $(-1,0] \cup (1,2)$, $[0,1/2) \cup (1,2]$, $A = \{x \in \mathbb{R} : |x| + 2x^2 > 1\}$.

Problem 1.3.16 Let μ be the counting measure on $\mathbb{R}$. Let us fix $A \subset \mathbb{R}$, and let us define $\nu(B) = \mu(B \cap A)$ for all $B \subset \mathbb{R}$.

a) If $A = \{1, 2, 3, \ldots, n, \ldots\}$ is ν a Borel-Stieltjes measure? If the answer is affirmative, find the distribution function.

b) And if $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$?

Problem 1.3.17 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\Phi : X \longrightarrow Y$ be a mapping. We define the *image space measure* $(Y, \mathcal{B}, \nu)$ as

$$\mathcal{B} = \Phi(\mathcal{A}) := \{B \subseteq Y : \ \Phi^{-1}(B) \in \mathcal{A}\}$$

and $\nu = \Phi(\mu) : \mathcal{B} \longrightarrow [0, \infty]$ given by $\nu(B) = \mu(\Phi^{-1}(B))$ for all $B \in \mathcal{B}$.
Prove that $(Y, \mathcal{B}, \nu)$ is a measure space and it is complete when $(X, \mathcal{A}, \mu)$ is.

Problem 1.3.18

a) Let $g : I \longrightarrow \mathbb{R}$ be a continuous and strictly increasing function. As g is injective it has a continuous inverse g^{-1}. Prove that $\mu_g = g^{-1}(m)$, that is to say that the Borel-Stieltjes measure with distribution function g coincides with the image measure of Lebesgue measure under g^{-1}.

b) Let $g : (0, \infty) \longrightarrow \mathbb{R}$ be the function $g(t) = \log t$. Prove that $\mu_g = g^{-1}(m) = e^m$ is invariant under dilations.

Hints: a) Prove that both measures coincide for semi-intervals $[a, b)$ and apply Caratheodory-Hopf's extension theorem. b) Use part a) and the fact that Lebesgue measure is translation invariant. Alternatively, it can be also proved by using Caratheodory-Hopf's extension theorem.

Problem 1.3.19 Let $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$ be the unit ball of $\mathbb{R}^n$ and $S_{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere. Let us consider the projection $\pi : B_n \setminus \{0\} \longrightarrow S_{n-1}$ given by $\pi(x) = x/\|x\|$. The $(n-1)$-dimensional Lebesgue measure on S_{n-1} is defined as $\sigma = n \cdot \pi(m)$, that is to say

$$\sigma(U) = n \cdot m(\pi^{-1}(U)), \qquad \text{for all } U \in \mathcal{B}(S_{n-1}).$$

Prove that σ is invariant under rotations.

Hint: Use problem 1.3.9.

Problem 1.3.20 Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let us consider the product set $X \times Y = \{(x, y) : \ x \in X, y \in Y\}$. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is the σ-algebra generated by the set $\mathcal{E} = \{A \times B : \ A \in \mathcal{A}, \ B \in \mathcal{B}\}$. Prove that there exists a unique measure $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \longrightarrow [0, \infty]$ such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B), \qquad \text{for all } A \in \mathcal{A}, \ B \in \mathcal{B}.$$

Hint: Prove that $\mathcal{E}$ is a semi-algebra and that $\mu \otimes \nu$ is countably additive on $\mathcal{E}$. Then apply Caratheodory-Hopf's extension theorem.