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Departamento de Matemáticas

# Integration and Measure. Problems

## Chapter 2: Integration theory

### Section 2.1: Integration of positive functions

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# 2 Integration Theory

## 2.1 Integration of positive functions

**Problem 2.1.1** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g : X \longrightarrow [0, \infty]$ be measurable positive functions and $A, B, E \in \mathcal{A}$, $\lambda \geq 0$. Prove that:

*i)* $\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu$ .

*ii)* $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

*iii)* $f \leq g \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

*iv)* $A \subseteq B \implies \int_A f \, d\mu \leq \int_B f \, d\mu$.

*v)* $\int_E f = 0 \Leftrightarrow f = 0$ a.e. in $E$.

*vi)* $\mu(E) = 0 \implies \int_E f \, d\mu = 0$.

*vii)* $A \cap B = \varnothing \implies \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$.

*viii)* $\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$.

*ix)* $f \geq g$, $\int_X g \, d\mu < \infty \implies \int_X f \, d\mu - \int g \, d\mu = \int_X (f - g) \, d\mu$.

*x)* $f \leq g$ a.e. in $E \implies \int_E f \, d\mu \leq \int_E g \, d\mu$.

*xi)* $f = g$ a.e. in $E \implies \int_E f \, d\mu = \int_E g \, d\mu$.

*Hints: vi*) If $f = 0$ a.e. and $s = \sum_j c_j \chi_{A_j} \leq f$, then $\mu(A_j) = 0$ for all $j$ and so $s = 0$ a.e. On the other hand, if $\mu(f > 0) > 0$, then $\mu(A) > 0$ for some $n \in \mathbb{N}$, where $A = \{f > 1/n\}$. Hence, $0 \leq s = \frac{1}{n} \chi_A \leq f$ and $\frac{1}{n} \mu(A \cap E) \leq \int_E f \, d\mu$, a contradiction. For the other statements, the idea is always to approximate positive functions by simple functions.

**Problem 2.1.2** Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $X = \cup_n X_n$, where $\{X_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of measurable subsets of $X$. Prove that if $f : X \longrightarrow [0, \infty]$ is a measurable positive function, then

$$\int_X f \, d\mu = \sum_n \int_{X_n} f \, d\mu \, .$$

*Hint:* Use the monotone convergence theorem.

**Problem 2.1.3** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow [0, \infty]$ be a measurable positive function. Let us define

$$\varphi(E) = \int_E f \, d\mu \, , \qquad \text{for all } E \in \mathcal{A}.$$

Prove that $\varphi$ is a measure on $\mathcal{A}$ and that

$$\int_X g \, d\varphi = \int_X g f \, d\mu \, , \qquad \text{for all } g : X \longrightarrow [0, \infty] \text{ measurable.} \tag{1}$$

*Note:* This fact justifies the notation $d\varphi = f \, d\mu$.

*Hint:* Apply Exercise 2.1.1 to prove that $\varphi$ is a measure. Then, prove (1) first for characteristic functions and simple functions and then approximate any positive function for a sequence of simple functions.

**Problem 2.1.4** Let $f : [0,1] \longrightarrow [0,\infty]$ be defined by $f(x) = 0$ if $x$ is rational, and otherwise $f(x) = n$ where $n$ is the number of zeros immediately after the decimal point in the representation of $x$ in the decimal scale. Calculate $\int f(x)\, dm$, being $m$ the Lebesgue measure.

*Hint:* $f(x) = k$ for $x \in [1/10^{k+1}, 1/10^k) \setminus \mathbb{Q}$.

**Problem 2.1.5** Let $f(x)$ be the function defined in $(0,1)$ by $f(x) = 0$ if $x$ is rational, and $f(x) = [1/x]$ if $x$ is irrational ($[t]$ denote the integer part of $t$). Decide whether or not $f$ is Lebesgue integrable and calculate $\int f(x)\, dm$ being $m$ the Lebesgue measure.

*Hint:* $f(x) = k$ for $x \in (1/(k+1), 1/k] \setminus \mathbb{Q}$.

**Problem 2.1.6** Let $(X, \mathcal{A}, \mu)$ be a probability space, i.e. $\mu(X) = 1$. Let $E \in \mathcal{A}$ be a set with $0 < \mu(E) < 1$. Put $f_n = \chi_E$ if $n$ is odd, $f_n = 1 - \chi_E$ if $n$ is even. What is the relevance of this example to Fatou's lemma?

*Hint:* $\liminf\limits_{n\to\infty} f_n(x) = 0$ for all $x \in X$ but $\liminf\limits_{n\to\infty} \displaystyle\int f_n\, d\mu = \min\{\mu(E), 1 - \mu(E)\}$.

**Problem 2.1.7** Let $f_{2n-1} = \chi_{[0,1]}$, $f_{2n} = \chi_{[1,2]}$, $n = 1, 2, \ldots$. Check that Fatou's Lemma is verified strictly for this sequence of functions.

*Hint:* $\liminf\limits_{n\to\infty} f_n(x) = 0$ for all $x \in \mathbb{R} \setminus \{1\}$ but $\liminf\limits_{n\to\infty} \displaystyle\int f_n\, dm = 1$.

**Problem 2.1.8**

a) Check that $\displaystyle\int_1^\infty \frac{1}{x}\, dm = \infty$, being $m$ the Lebesgue measure.

b) Let $p \in \mathbb{R}$. Prove that:

  b1) $\displaystyle\int_0^\infty e^{-px} dm < \infty$ if and only if $p > 0$.

  b2) $\displaystyle\int_1^\infty \frac{1}{x^p}\, dm < \infty$ if and only if $p > 1$.

  b3) $\displaystyle\int_0^1 \frac{1}{x^p}\, dm < \infty$ if and only if $p < 1$.

*Hint:* a) $\dfrac{1}{x} = \lim\limits_{N\to\infty} \dfrac{1}{x} \chi_{[1,N]}(x)$. Apply the monotone convergence theorem.

**Problem 2.1.9** Prove that the function $f(x) = \frac{1}{\sqrt{x}}$ if $x \in (0,1]$, and $f(0) = 0$, is Lebesgue-integrable in $[0,1]$ and calculate its integral.

*Hint:* $f$ is almost everywhere continuous and $f(x) = \lim\limits_{\varepsilon\to 0^+} \frac{1}{\sqrt{x}} \chi_{[\varepsilon,1]}(x)$ if $x \in [0,1]$.

**Problem 2.1.10** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow [0, \infty]$ be a measurable positive function. Let $f_n(x) = \min\{f(x), n\}$. Prove that $\displaystyle\int_X f_n \, d\mu \nearrow \int_X f \, d\mu$.

*Hint:* Use an adequate convergence theorem.

**Problem 2.1.11** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow [0, \infty]$ be a sequence of measurable positive functions. Let us suppose that $\exists \lim_{n\to\infty} f_n = f$ and that $f_n \leq f$ for all $n$. Prove that $\displaystyle\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.

*Hint:* Use Fatou's Lemma and $\int_X f_n \, d\mu \leq \int_X f \, d\mu$.

**Problem 2.1.12**

a) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow [0, \infty]$ be a sequence of measurable positive functions. Let us suppose that $f_n(x) \searrow f(x)$ and that $\int_X f_k \, d\mu < \infty$ for some $k$. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

b) Let $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$ and $a > 0$. Let us define $f_n(x) = a_n/x$, for $x > a > 0$. Check that $f_n$ decreases uniformly to 0 but $\int f_n \, dm = \infty$ for all $n$.

*Hint:* a) Consider the sequence $g_n = f_k - f_{k+n}$.

**Problem 2.1.13** Let $g : (X, \mathcal{A}, \mu) \longrightarrow [0, \infty]$ be an integrable function. Let $\{E_n\}$ be a decreasing sequence of sets such that $\cap_{n=1}^{\infty} E_n = \varnothing$. Prove that $\lim_{n\to\infty} \int_{E_n} g \, d\mu = 0$.

**Problem 2.1.14** Prove that for all $a > 0$, the function $f(x) = e^{-x} x^{a-1}$ is Lebesgue-integrable in $[0, \infty]$.

**Problem 2.1.15** Let $f_n : [0, 1] \longrightarrow [0, \infty)$ be a sequence of positive functions defined by

$$f_n(x) = \begin{cases} n \,, & \text{if } 0 \leq x \leq 1/n \,, \\ 0 \,, & \text{otherwise.} \end{cases}$$

Check that $f_n \to 0$ pointwise when $x > 0$ but $\int f_n \, dm = 1$. Interpret why this may happen.

**Problem 2.1.16** Let $\mathcal{M}$ be the $\sigma$-algebra of Lebesgue-measurable sets in $[0, \infty)$. We define in $\mathcal{M}$ the measure $\mu$ as

$$\mu(E) = \int_E \frac{1}{1+x} \, dx \,.$$

Check that $\mu$ is a Borel-Stieltjes measure and calculate the corresponding distribution function $F$. Find a function $f(x)$ such that $\int f \, d\mu < \infty$ but $\int f \, dm = \infty$, being $m$ the Lebesgue measure.

*Hint:* $F(t) = \log(1 + t) \, \chi_{[0,\infty]}(t)$; $f(x) = 1/(1 + x)$.

**Problem 2.1.17** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces and let $A, A_i \in \mathcal{A}$, $B, B_i \in \mathcal{B}$ $(i \in \mathbb{N})$ be sets such that

$$A \times B = \bigcup_{i=1}^{\infty} (A_i \times B_i)\,, \qquad A_i \times B_i \text{ disjoint sets.}$$

Prove that

$$\mu(A)\nu(B) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)\,.$$

*Hint:* Use that for a positive sequence of functions: $\sum_n \int f_n = \int \sum_n f_n$.

**Problem 2.1.18** Prove Borel-Cantelli Lemma (see Problem 1.2.11) using the the monotone convergence theorem.

*Hint:* Consider the function $\sum_{n=1}^{\infty} \chi_{A_n}$.

**Problem 2.1.19** Let $A = [0,1] \cap \mathbb{Q}$. Then we can write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. Let us define the functions $f_n : [0,1] \longrightarrow \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1\,, & \text{if } x \in \{a_1, \ldots, a_n\}\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Prove that $f_n$ is Riemann-integrable and calculate $f(x) = \lim_{n\to\infty} f_n(x)$. Are $f_n$ and $f$ Lebesgue-integrable functions?

**Problem 2.1.20** With the notation of the problem above, let $F(x)$ be the function

$$F(x) = \begin{cases} \frac{1}{k}\,, & \text{if } x = a_k\,, \\ 0\,, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

Show that the function $F$ is Riemann-integrable on any bounded interval $[a, b]$ and find $\int_a^b F(x)\,dx$.