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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 2: Integration theory

Section 2.2: Integration of general functions

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2 Integration Theory

2.2. Integration of general functions

Problem 2.2.1 Let $f_n : [0,1] \longrightarrow [-1,1]$ be a sequence of continuous functions such that $f_n(x) \to 0$ almost everywhere with respect to Lebesgue measure. Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0\,.$$

Hint: The functions f_n are uniformly bounded.

Problem 2.2.2 Let $(X, \mathcal{A}, \mu)$ be a finite space measure: $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n(x) \to f(x)$ uniformly in X. Prove that $f \in L^1(\mu)$ and that

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu\,.$$

Hint: Uniform convergence implies that the sequence f_n is uniformly-Cauchy.

Problem 2.2.3* Let $f_n : (\mathbb{R}, \mathcal{M}, m) \longrightarrow [0,\infty)$ be a sequence of positive Lebesgue-measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost all $x \in \mathbb{R}$ and, besides, $\int_{\mathbb{R}} f_n\,dx = \int_{\mathbb{R}} f\,dx = 1$ for all $n \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{x} f_n\,dx = \int_{-\infty}^{x} f\,dx\,, \qquad \text{for all } x \in \mathbb{R}\,.$$

Hint: Consider the functions $\min(f_n, f)$ and use an adequate convergence theorem. Recall that $\min(x,y) = \frac{x+y-|x-y|}{2}$.

Problem 2.2.4 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \longrightarrow \mathbb{R}$ be an integrable function.

a) Prove Markov's inequality:

$$\mu(\{x \in X :\ |f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon} \int_X |f|\,d\mu\,.$$

b) Using Markov's inequality, show that if f is a measurable function, then

$$\begin{array}{lllll} \text{b1)} & \int |f|\,d\mu & = 0 & \Longleftrightarrow & \mu(f \neq 0) = 0\,, \\ \text{b2)} & \int |f|\,d\mu & < \infty & \Longrightarrow & \mu(|f| = \infty) = 0\,. \end{array}$$

Give an example showing that it is possible to have that

$$\int |f|\,d\mu = \infty \qquad \text{and} \qquad \mu(|f| = \infty) = 0\,.$$

Hints: a) $1 \leq \frac{1}{\varepsilon}|f|$ on the set $\{x \in X : |f(x)| \geq \varepsilon\}$. b1) If $\int |f|\,d\mu = 0$, then $\mu(|f(x)| \geq 1/n)\}) = 0$ for all $n \in \mathbb{N}$. b2) If $\int |f|\,d\mu < \infty$, then $\{|f| = \infty\} \subset \{|f| \geq n\}$ for all $n \in \mathbb{N}$.

Problem 2.2.5 Prove that the function $f(x) = \dfrac{\sin x}{x}$ is not Lebesgue-integrable in $(0,\infty)$.

Hint: Divide $(0,\infty)$ in the intervals $(n\pi, (n+1)\pi]$ $(n \geq 0)$.

Problem 2.2.6 Discuss whether the following functions are Lebesgue integrable or not. Give an argument of why they are not, or find the value of the integral:

a) $f(x) = \dfrac{1-\cos x}{x(1+x^2)}$ for $x \in (0,\infty)$.

b) $g(x) = \sin x + \cos x$ for $x \in \mathbb{R}$.

Hints: a) On $(0,\delta)$ we have $|f(x)| \le Cx/(1+x^2) \in L^1(0,\delta)$ and on (δ,∞) we have $|f(x)| \le 2/x^3 \in L^1(\delta,\infty)$. b) $|\sin x + \cos x|$ is π-periodic, $f(x) > 0$ on $(-\pi/4, 3\pi/4)$ and $\int_{-\pi/4}^{3\pi/4} |\sin x + \cos x|\, dx = 2\sqrt{2} > 0$.

Problem 2.2.7 It is easy to guess the limits

a) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx\,,$

b) $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx\,.$

Prove that your guesses are correct.

Problem 2.2.8 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_n : X \longrightarrow \mathbb{R}$ be a sequence of measurable functions such that

$$\sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty\,.$$

Prove that:

a) The series $\sum_n f_n$ converges almost everywhere in X to a function $f : X \longrightarrow \mathbb{R}$:

$$\sum_{n=1}^{\infty} f_n(x) = f(x)\,, \qquad \text{for almost every } x \in X\,.$$

b) $f \in L^1(\mu)$.

c) $\displaystyle\int_X f\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu\,.$

Hints: a) Consider the function $F(x) := \sum_{n=1}^{\infty} |f_n(x)| \in L^1(X)$, why? Then $|f(x)| \le F(x) < \infty$ almost everywhere (use problem 2.2.4). b) It follows easily from a). c) $g_n := f_1 + \cdots + f_n$ verifies $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. and $|g_n| \le F$. Use a convergence theorem.

Problem 2.2.9 Prove that

$$\lim_{n\to\infty} \int_0^{\infty} \frac{dx}{(1+x/n)^n x^{1/n}} = 1\,.$$

Hint: $f_n(x) \le \frac{1}{\sqrt{x}} \chi_{(0,1]}(x) + (1+x/2)^{-2} \chi_{(1,\infty)}(x) \in L^1(0,\infty)$ for $n \ge 2$ and so we can use dominated convergence.

Problem 2.2.10 Let us consider the functions

$$f_n(x) = \frac{nx-1}{(x\log n + 1)(1 + nx^2 \log n)}\,, \qquad x \in (0,1]\,.$$

Prove that

$$\lim_{n\to\infty} f_n(x) = 0\,, \qquad \text{but} \quad \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2}\,.$$

What is the relevance of this result?

Hint: Prove that $\dfrac{nx-1}{(x\log n+1)(1+nx^2\log n)} = \dfrac{-1}{x\log n+1} + \dfrac{nx}{(n\log n)x^2+1}$.

Problem 2.2.11 Consider $a>0$.

a) Prove that for each $x \geq a$ the function $v(t) := \dfrac{t}{1+t^2x^2}$ decreases for $t \geq 1/a$.

b) Find an upper bound of the function

$$f_n(x) = \frac{n}{1+n^2x^2}\,, \qquad x\geq a\,,\ n \geq 1/a\,,$$

by a function which just depends on x and a.

c) Calculate

$$L = \lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx\,,$$

and say what theorem you used.

d) Calculate L using monotone convergence theorem and Barrow's rule in the cases $a>0$, $a=0$, $a<0$.

Problem 2.2.12 Calculate $L = \displaystyle\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2}{(1+x^2)^n}\,dx$.

Hint: $\{f_n\}$ is a decreasing sequence and $f_2 \in L^1((0,\infty))$. So we can use a convergence theorem.

Problem 2.2.13 Prove that $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{\log(n+x)}{n}\,e^{-x}\cos x\,dx = 0$.

Problem 2.2.14 Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be the sequence of measurable functions defined by

$$f_n(x) = \begin{cases} n\cos nx\,, & \text{if } x \in [-\frac{\pi}{2n}, \frac{\pi}{2n}]\,, \\ 0\,, & \text{otherwise.}\end{cases}$$

Study whether

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f_n(x)\,dx = \int_{-\pi}^{\pi}\lim_{n\to\infty} f_n(x)\,dx$$

or not. Can be applied in this case the monotone convergence theorem or the Lebesgue dominated convergence theorem?

Problem 2.2.15 Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ the measure space defined by

$$\mu(A) = \operatorname{card}(A\cap\mathbb{N})\,, \qquad A \in \mathcal{B}(\mathbb{R})\,.$$

Prove that $f(x) = x\sin(\pi x)$ is μ-integrable but not Lebesgue-integrable.

Problem 2.2.16

a) Prove that the sequence of functions

$$f_n(t) = \left(1 + \frac{t}{n}\right)^n, \qquad t \geq 0,$$

verify that $f_3(t) \leq f_n(t)$ for $n \geq 3$.

b) Calculate

$$\lim_{n\to\infty} \int_0^1 \frac{n + n^2 x}{(1+x)^n}\, dx\,.$$

State correctly the results and theorems you need to get to the solution.

Hint: b) To start, do the change of variable $t = nx$.

Problem 2.2.17 Calculate $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{n + n^4 x^3}{(1+x)^n}\, dx\,.$

Problem 2.2.18 Prove that $\displaystyle\int_0^1 \frac{x}{1-x} \log\frac{1}{x}\, dx = \sum_{n=2}^{\infty} \frac{1}{n^2}\,.$

Hint: Use that $1/(1-x) = \sum_{n=0}^{\infty} x^n$ for $x \in (0,1)$ and then apply an adequate convergence theorem.

Problem 2.2.19 Let $(X, \mathcal{P}(X), \mu)$ be a measure space with X countable, $X = \{x_n\}_{n=1}^{\infty}$, and μ the discrete measure defined as:

$$\mu(\{x_n\}) = p_n\,, \qquad \mu(A) = \sum_{x_n \in A} p_n\,, \qquad (p_n \geq 0)\,.$$

Let $f : X \longrightarrow \mathbb{C}$ be a complex function.

a) Prove that if $f \geq 0$, then $\displaystyle\int_X f\, d\mu = \sum_{n=1}^{\infty} f(x_n)\, p_n\,.$

b) Prove that $f \in L^1(\mu)$ if and only if $\sum_{n=1}^{\infty} |f(x_n)|\, p_n < \infty$, and in this case,

$$\int_X f\, d\mu = \sum_{n=1}^{\infty} f(x_n)\, p_n\,.$$

Hints: a) $f = \sum_{n=1}^{\infty} f(x_n)\, \chi_{\{x_n\}}$. b) Decompose $f = u + iv$ and $u = u^+ - u^-$, $v = v^+ - v^-$.

Problem 2.2.19 Calculate $\displaystyle\lim_{n\to\infty} n \sum_{i=1}^{\infty} \sin\frac{2^{-i}}{n}\,.$

Hint: Consider and adequate measure space and apply a convergence theorem.

Problem 2.2.21* Let $(X, \mathcal{A}, \mu)$ be a measure space and $\Phi : X \longrightarrow Y$ be a mapping. Let us consider the image measure space $(Y, \mathcal{B}, \nu)$ by Φ ($\mathcal{B} = \Phi(\mathcal{A})$ and $\nu = \mu \circ \Phi^{-1}$). Let $f : Y \longrightarrow \mathbb{C}$ be a function. Prove that

a) f is $\mathcal{B}$-measurable if and only if $f \circ \Phi$ is $\mathcal{A}$-measurable .

b) If $f \geq 0$ is $\mathcal{B}$-measurable, then $\displaystyle\int_Y f\, d\nu = \int_X (f \circ \Phi)\, d\mu$.

c) If f is $\mathcal{B}$-measurable, then $f \in L^1(\nu)$ if and only if $f \circ \Phi \in L^1(\mu)$, and in this case

$$\int_Y f\, d\nu = \int_X (f \circ \Phi)\, d\mu.$$

d) Let $\Phi(x, y) = x^2 y$ be defined on the square $Q = [0,1] \times [0,1]$ in the plane, and let m be two-dimensional Lebesgue measure on Q. If μ is the image measure of m by Φ, evaluate the integral $\int_{-\infty}^{\infty} t^2\, d\mu(t)$.

Hints: a) Use the definition of $\mathcal{A}$. b) Prove it first for simple functions and then approximate any $f \geq 0$ by simple functions and apply monotone convergence. c) Decompose $f = u + iv$ and $u = u^+ - u^-$, $v = v^+ - v^-$. d) Apply c).

Problem 2.2.22 Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\rho : X \longrightarrow [0, \infty]$ be a measurable function. Let us consider the measure defined by the density ρ:

$$\nu(A) = \int_A \rho\, d\mu\,, \qquad A \in \mathcal{A}.$$

Prove that

a) If $f \geq 0$ is measurable, then $\displaystyle\int_X f\, d\nu = \int_X f \rho\, d\mu$.

b) If f is measurable, then: $f \in L^1(\nu)$ if and only if $\displaystyle\int_X |f| \rho\, d\mu < \infty$, and in this case

$$\int_X f\, d\nu = \int_X f \rho\, d\mu.$$

Hints: a) This is the exercise 2.1.3. b) Decompose $f = u + iv$ and $u = u^+ - u^-$, $v = v^+ - v^-$.

Problem 2.2.23 Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and $dm = dx\, dy$ be the Lebesgue measure on X. Let $\Phi : X \longrightarrow \mathbb{R}$ be the function given by $\Phi(x, y) = \log(x^2 + y^2)$ and let μ be the image measure of dm by Φ.

a) Calculate the value of $\mu([0,1])$.

b) Prove that μ has the form $d\mu = F(t)\, dt$ and find $F(t)$ explicitly.

Hints: a) $\mu([0,1]) = m(\{(x, y) : 0 \leq \log(x^2 + y^2) \leq 1\})$. b) Calculate $\int_{\mathbb{R}} f(t) d\mu(t)$ for any $f \in L^1(\mu)$.

Problem 2.2.24

a) Let $f : \mathbb{R} \to [0, \infty]$ be an integrable function on $\mathbb{R}$ and such that $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Prove that $F(x) = \int_{-\infty}^{x} f(y) dy$ is a probability distribution function and that besides F is continuous (f is called the density function).

b) Prove that the Borel-Stieltjes measure with distribution function F coincides with the measure defined with the density function f: $\nu_f(A) = \int_A f(x)\, dx$.

c) Calculate $F(x)$ if

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Hints: a) F is increasing because $f \geq 0$ and is continuous by the dominated convergence theorem. b) The Borel-Stieltjes measure μ_F coincides with the density measure ν_f by the Caratheodory-Hopf's extension theorem since for semi-intervals $[a,b)$ we have: $\mu_F([a,b)) = F(b) - F(a) = \int_a^b f(x)\,dx = \nu_f([a,b))$. Observe that $\mu_F(\{a\})=0$ for all $a \in \mathbb{R}$ since F is continuous. c) $F(x) = 0$, if $x \leq 0$, $F(x) = x$ if $x \in [0,1]$ and $F(x) = 1$ if $x \geq 1$.

Problem 2.2.25* Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing derivable function with bounded derivative g' on each compact set. Let us consider the Borel-Stieltjes measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m_g)$. Prove that $m_g = g'dm$, that is to say that the Borel-Stieltjes measure m_g coincides with the measure defined by the density g' and therefore for all $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f \in L^1(m_g)$, we have

$$\int_{\mathbb{R}} f\,dm_g = \int_{\mathbb{R}} fg'\,dm = \int_{\mathbb{R}} f(t)\,g'(t)\,dt\,.$$

Hint: Use the Caratheodory-Hopf extension theorem and that $\int_a^b g'\,dm = g(b) - g(a)$. This is trivial if g' is continuous by Barrow's rule, but for g' only bounded we must use an approximation argument: let $g_n(t) = (f(t+h_n) - f(t)/h_n$. Then $g_n \longrightarrow g'$ for all $t \in [a,b)$. Use dominated convergence to conclude that $\int_a^c g'\,dm = g(c) - g(a)$ for all $c \in [a,b)$. Finally use monotone convergence, since $[a,b) = \cup_n [a, c_n]$ with $c_n \nearrow b$ as $n \to \infty$.

Problem 2.2.26* Let us consider the Lebesgue measure space $(\mathbb{R}^n, \mathcal{M}, m)$, where $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets and m is Lebesgue measure. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function. Prove that

a) If $f \geq 0$ or if $f \in L^1(m)$, then

a.1) $\displaystyle\int_{\mathbb{R}^n} f(a+x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx\,.$

a.2) $\displaystyle\int_{\mathbb{R}^n} f(T(x))\,dx = \frac{1}{|\det T|}\int_{\mathbb{R}^n} f(x)\,dx$, for all $T \in GL(n)$.

a.3) More generally, $\displaystyle\int_A f(T(x))\,dx = \frac{1}{|\det T|}\int_{T(A)} f(x)\,dx$, for all $T \in GL(n)$ and $A \in \mathcal{M}$.

b) If $\Phi : \mathbb{R} \longrightarrow [0,\infty]$ is a Borel function, then

$$\int_{\mathbb{R}^n} \Phi(\|x\|)\,dx = n\Omega_n \int_0^\infty \Phi(r)\,r^{n-1}\,dr\,, \qquad \text{where } \Omega_n = m(\{x \in \mathbb{R}^n : \ \|x\| \leq 1\})\,.$$

c) Let $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$. Then

$$\int_{B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha < n \qquad \text{and} \qquad \int_{\mathbb{R}^n \setminus B_n} \frac{dx}{\|x\|^\alpha} < \infty \quad \Leftrightarrow \quad \alpha > n\,.$$

Hints: Let $\mu = T(m)$ be the image measure of m under T: a.1) If $T(x) = a+x$, then $\mu(A) = m(A)$ since m is translation-invariant. a.2) $\mu(A) = m(T^{-1}(A)) = |\det T^{-1}|\,m(A)$. This fact is easy for semi-intervals $[a_1, b_1) \times \cdots \times [a_n, b_n)$ and so it is a consequence of Caratheodory-Hopf extension theorem. a.3) It follows from a.2) and the fact that, as T is bijective, we have $\chi_{T(A)} \circ T = \chi_A$. b) Let $\nu = \| \cdot \| \circ m$ be the image measure under $\| \cdot \|$: then prove that $\nu[a, b) = \Omega_n(b^n - a^n)$ and as $g(t) = \Omega_n t^n$ is increasing and continuous, conclude from Exercise 2 that $\nu = g' \, dm = n\Omega_n t^{n-1} dt$. c) Apply part b).

Problem 2.2.27* Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue-integrable function. Evaluate

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n) \, \frac{x}{1+|x|} \, dx \, .$$

Hint: Apply the change of variables $y = x + n$ and divide the integral in two parts: one on the interval $(-\infty, -n)$ and the other one on $(-n, \infty)$. Apply Lebesgue dominated convergence theorem to prove that the first integral converges to 0 and the second one to $\int_{-\infty}^{\infty} f(x) \, dx$.