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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.1: Continuity and differentiability

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3 Parametric integrals

3.1 Continuity and differentiability

Problem 3.1.1 Let $f(x,y) = \log(x^2+y^2)$ for $y \in (0,1)$ and $x > 0$.

a) Prove that $\varphi(x) = \int_0^1 f(x,y)\,dy$ is well defined and is derivable. Prove that $\varphi'(x) = \int_0^1 \frac{\partial f}{\partial x}\,dy$ and calculate $\varphi'(x)$.

b) Prove that $\varphi(x)$ is continuous at $x_0 = 0$ and that $\varphi(0) = -2$.

c) Compute $\varphi(x)$ integrating by parts.

Hint: $f(x,\cdot)$ is continuous on $[0,1]$ for fixed $x > 0$. Besides $\left|\frac{\partial}{\partial x}\big[f(x,y)\big]\right| \le \frac{2}{x_0} \in L^1(0,1)$ for $x \ge x_0 > 0$. Hence, F is derivable on (x_0,∞) for all $x_0 > 0$ and so it is derivable on $(0,\infty)$.

Problem 3.1.2 Let $F, G : \mathbb{R} \longrightarrow \mathbb{R}$ defined as

$$F(x) = \left(\int_0^x e^{-t^2} dt\right)^2 \qquad \text{and} \qquad G(x) = \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}\,dt\,.$$

Prove that:

a) $F'(x)+G'(x) = 0$, for all $x \in \mathbb{R}$. Justify why you can apply the theorem on differentiation of parametric integrals.

b) $F(x)+G(x) = \pi/4$, for all $x \in \mathbb{R}$.

c) Deduce that $\displaystyle\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Hints: a) $\left|\frac{\partial}{\partial x}\big[\frac{e^{-x^2(1+t^2)}}{1+t^2}\big]\right| = |2xe^{-x^2(1+t^2)}| \le 2 \in L^1[0,1]$ for $x \in \mathbb{R}$. c) Let $x \to \infty$ in b) by applying monotone convergence.

Problem 3.1.3 Calculate $F(s) = \displaystyle\int_0^\infty e^{-x}\sin(sx)\,dx$, and, justifying all the steps, from the obtained result calculate

$$G(s) = \int_0^\infty x\,e^{-x}\cos(sx)\,dx\,.$$

Hints: Use integration by parts to evaluate $F(s)$; $G(s)$ is derivable since $\left|\frac{\partial}{\partial s}\big[e^{-x}\sin(sx)\big]\right| \le x\,e^{-x} \in L^1(0,\infty)$.

Problem 3.1.4

a) Assuming that we can apply the Fundamental Theorem of Calculus and the theorem on parametric derivation, prove that:

$$F(x) = \int_a^{f(x)} g(x,t)\,dt \qquad \Longrightarrow \qquad F'(x) = g(x,f(x))\,f'(x) + \int_a^{f(x)} \frac{\partial g}{\partial x}(x,t)\,dt\,.$$

b) Prove that

$$\int_0^{\pi/(4a)} \frac{x}{\cos^2 ax}\,dx = \frac{1}{2a^2}\left(\frac{\pi}{2} - \log 2\right), \qquad \text{for } a > 0.$$

Hints: a) Consider the function $G(u,v) = \int_a^v g(u,t)\,dt$ and apply the chain rule. b) Use the previous part to calculate the derivative of $\int_0^{\pi/(4a)} \tan ax\,dx$ with respect to a.

Problem 3.1.5 Prove that

$$J(a) = \int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{\pi+2}{8a^3}\,, \qquad \text{for } a>0.$$

Hint: $\left|\frac{\partial}{\partial a}\left[\frac{1}{x^2+a^2}\right]\right| = \frac{2a}{(x^2+a^2)^2} \le \frac{2M}{(x^2+\varepsilon^2)^2} \in L^1(0,\infty)$ for $a \in [\varepsilon, M]$.

Problem 3.1.6 Let $F(\alpha) = \displaystyle\int_0^\infty \frac{e^{-\alpha x}-e^{-x}}{x}\,dx.$

a) Study when the integral converges.

b) Calculate $F'(\alpha)$ explicitly and then calculate $F(\alpha)$.

c) Obtain the successive derivatives $F^{(k)}(\alpha)$ and calculate $\displaystyle\int_0^\infty x^n e^{-x} dx.$

Hints: a) $\lim_{x\to 0^+} \frac{e^{-\alpha x}-e^{-x}}{x} = 1-\alpha$ and so, $\int_0^1 \frac{e^{-\alpha x}-e^{-x}}{x}\,dx < \infty$. Also, $\int_1^\infty \left|\frac{e^{-\alpha x}-e^{-x}}{x}\right| dx \le \int_0^\infty (e^{-\alpha x}+e^{-x})\,dx < \infty$ if $\alpha > 0$. b) $\left|\frac{\partial}{\partial \alpha}\left[\frac{e^{-\alpha x}-e^{-x}}{x}\right]\right| \le e^{-\alpha_0 x} \in L^1(0,\infty)$ for $\alpha > \alpha_0 > 0$ and so F is derivable on (α_0,∞) for all $\alpha_0 > 0$. c) Derive both members of the identity $F'(\alpha) = -\int_0^\infty e^{-\alpha x} dx = -1/\alpha$.

Problem 3.1.7 Prove that for $a > 0$ and $b > 0$:

$$F(a,b) = \int_0^\infty (e^{-a^2/x^2} - e^{-b^2/x^2})\,dx = \sqrt{\pi}(b-a)\,.$$

Hint: $\left|\frac{\partial}{\partial a}\left[e^{-a^2/x^2} - e^{-b^2/x^2}\right]\right| \le \frac{2a}{x^2} e^{-a_0^2/x^2} \in L^1(0,\infty)$ for $a \ge a_0 > 0$. Hence, F is derivable on $[a_0,\infty)$ for all $a_0 > 0$ and so it is derivable on $(0,\infty)$. To compute $\frac{\partial}{\partial a}F(a,b)$ change variables to $t = 1/x$. Recall that $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$ and observe that $F(a,a) = 0$.

Problem 3.1.8 Explain in the following cases why we can differentiate the parametric integral and why they are well-defined. Obtain explicitly the function deriving with respect to the parameter and integrating later with respect to it:

i) $F(s) = \displaystyle\int_0^{\pi/2} \log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{dx}{\cos x}$, with $|s| < 1$.

ii) $G(a) = \displaystyle\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right) dx$, with $a \in \mathbb{R}$.

iii) $H(p) = \displaystyle\int_0^1 \frac{x^p - 1}{\log x}\,dx$, with $p > -1$.

iv) $I(\lambda) = \displaystyle\int_0^{\pi/2} \frac{\log(1-\lambda^2 \sin^2 x)}{\sin x}\,dx$, with $|\lambda| < 1$.

v) $K(x) = \displaystyle\int_0^\infty e^{-t^2 - x^2/t^2} dt$, with $x \in \mathbb{R}$.

Hints: i) $\left|\frac{\partial}{\partial s}\left[\log\left(\frac{1+s\cos x}{1-s\cos x}\right)\frac{1}{\cos x}\right]\right| \leq \frac{2}{1-s_0^2\cos^2 x} \in L^1(0,\pi/2)$ if $|s| \leq s_0 < 1$. *ii)* Since G is an even function, it is enough to consider the case $a \geq 0$; $\left|\frac{\partial}{\partial a}\left[\log(1+\frac{a^2}{x^2})\right]\right| = \frac{2|a|}{x^2+a^2} \leq \frac{2M}{x^2+\varepsilon^2} \in L^1(0,\infty)$ if $|a| \in [\varepsilon, M]$. *iii)* $\left|\frac{\partial}{\partial p}\left[\frac{x^p-1}{\log x}\right]\right| = x^p \in L^1(0,1)$ since $p > -1$. *iv)* $\left|\frac{\partial}{\partial \lambda}\left[\frac{\log(1-\lambda^2\sin^2 x)}{\sin x}\right]\right| = \frac{2|\lambda||\sin x|}{1-\lambda^2\sin^2 x} \leq \frac{2}{1-\lambda_0\sin^2 x} \in L^1(0,\pi/2)$ if $|\lambda| < \lambda_0 < 1$. *v)* $\left|\frac{\partial}{\partial x}\left[e^{-t^2-x^2/t^2}\right]\right| \leq \frac{2M}{t^2}\,(e^{-t^2}\chi_{[1,\infty)}(t) + e^{-\varepsilon^2/t^2}\chi_{(0,1)}(t)) \in L^1(0,\infty)$ if $|x| \in [\varepsilon, M]$. To compute $K'(x)$ change variables to $s = x/t$ and prove that $K'(x) = -2K(x)$. Note that $K(x)$ is even and so it is enough to compute it for $x \geq 0$.

Problem 3.1.9 Obtain explicitly the function $F(t)$ justifying all the steps:

$$F(t) = \int_0^\infty e^{-tx}\,\frac{\sin x}{x}\,dx\,, \qquad \forall\, t > 0\,.$$

Hint: As $\left|\frac{\partial}{\partial t}\left[e^{-tx}\frac{\sin x}{x}\right]\right| \leq e^{-tx} \leq e^{-\varepsilon x} \in L^1(0,\infty)$ for $t \in (\varepsilon,\infty)$, we have that $F(t)$ is derivable on (ε,∞) for all $\varepsilon > 0$ and so it is derivable on $(0,\infty)$.

Problem 3.1.10 Prove that

$$\int_0^\infty \frac{1-\mathrm{e}^{-x^2}}{x^2}\,dx = \sqrt{\pi}.$$

Hint: Consider the function $F(t) = \int_0^\infty \frac{1-\mathrm{e}^{-tx^2}}{x^2}\,dx$ for $t > 0$ and proceed in a similar way to the previous problems.

Problem 3.1.11 Let $F(\lambda) = \displaystyle\int_0^\infty \frac{dx}{x^2+\lambda}$. Write the derivatives of F, and later prove that for all $\lambda > 0$,

$$\int_0^\infty \frac{dx}{(x^2+\lambda)^{n+1}} = \frac{1\cdot 3\cdots(2n-1)}{2^n n!}\,\frac{\pi}{2\lambda^{n+1/2}} = \frac{(2n)!\,\pi}{(n!)^2(2\sqrt{\lambda}\,)^{2n+1}}\,.$$

Hints: First of all, it is easy to calculate $F(\lambda)$ and then all its derivatives $F^{(n)}(\lambda)$. Also, $\left|\frac{\partial}{\partial \lambda}\left[\frac{1}{x^2+\lambda}\right]\right| = \frac{1}{(x^2+\lambda)^2} \leq \frac{1}{(x^2+\lambda_0)^2} \in L^1(0,\infty)$ for $\lambda > \lambda_0 > 0$. Hence, F is derivable on (λ_0,∞) for all $\lambda_0 > 0$ and so it is derivable on $(0,\infty)$. Similarly, we can see that F is infinitely derivable on $(0,\infty)$, and its derivatives can be calculated by parametric derivation: $F^{(n)}(\lambda) = \int_0^\infty \frac{\partial^n}{\partial\lambda^n}\left[\frac{1}{x^2+\lambda}\right]dx$.

Problem 3.1.12 Let

$$F(x) = \int_0^{2x} \frac{\log(1+2xt)}{1+t^2}\,dt\,, \qquad x \geq 0\,.$$

a) Check that F is derivable on $(0,\infty)$ and prove that

$$F'(x) = \frac{\log(1+4x^2)}{1+4x^2} + \frac{4x}{1+4x^2}\ \arctan 2x\,.$$

b) Using the previous part, prove that

$$F(x) = \log\sqrt{1+4x^2}\ \arctan 2x\,.$$

Hints: a) $\left|\frac{\partial}{\partial x}\left[\frac{\log(1+2xt)}{1+t^2}\right]\right| \leq \frac{2t}{(1+t^2)(1+2x_0t)} \in L^1(0,\infty)$, for $x > x_0 > 0$. Hence, F is derivable on (x_0,∞) for all $x_0 > 0$ and so it is derivable on $(0,\infty)$. To calculate $F'(x)$ use decomposition on simple fractions. b) Integrate by parts.

Problem 3.1.13* Prove that

$$\int_0^\pi \frac{\log(1+\cos x)}{\cos x}\,dx = \frac{\pi^2}{2},$$

calculating first

$$F(t) := \int_0^\pi \frac{\log(1+t\cos x)}{\cos x}\,dx \qquad \text{for } |t| \leq 1\,.$$

Hints: $\left|\frac{\partial}{\partial t}\left[\frac{\log(1+t\cos x)}{\cos x}\right]\right| = \frac{1}{1+t\cos x}$ which is continuous for $|t| < 1$, and so it belongs to $L^1(0,\pi)$. This means that $F(t)$ is derivable on $(-1,1)$. Compute $F(t)$ by using parametric derivation and calculate $F'(t) = \pi/\sqrt{1-t^2}$ (change variables to $u = \tan(x/2)$). Now, if $0 \leq t \leq 1$, we have that $f(x,t) = \frac{\log(1+t\cos x)}{\cos x}$ verifies, for $x \in [0,\pi/2)$, that $f(x,t) \leq \frac{\log(1+\cos x)}{\cos x}$ which is continuous at $x = \pi/2$ and so it belongs to $L^1[0,\pi/2)$, and for $x \in (\pi/2,\pi)$ that $f(x,t) \leq g(x) := \frac{1}{|\cos x|}\log\frac{1}{1-|\cos x|}$. But $g(x)$ is continuous at $x = \pi/2$ and $\log\frac{1}{1-|\cos x|} \in L^1[\pi/2,\pi)$ since $\lim_{x\to\pi^-}\frac{\log(1+\cos x)}{(\pi-x)^{-\varepsilon}} = 0$ for each $\varepsilon > 0$. Hence, $F(t)$ is continuous on $[0,1]$ and $F(1) = \lim_{t\to 1^-} F(t)$.

Problem 3.1.14* Let us consider the function

$$F(x) = \int_0^1 \frac{(\log(1-xt))^2}{t}\,dt.$$

a) Find the values of x such that $F(x)$ is defined.

b) Calculate $F'(x)$ justifying why you can derive. Evaluate the resulting integral.

c) Study the increasing and decreasing intervals of F.

Hints: a) As $\lim_{z\to 0^+}(\log(1-z))/z = 1$ we have $\log(1-z) \leq Cz$ for $0 < z < \delta$. As $\lim_{z\to 0^+} z^\varepsilon \log z = 0$ we have $|\log z| \leq z^{-\varepsilon}$ for $0 < z < \delta'$. b) If $x < x_0 < 1$, then $\frac{\partial}{\partial x}\left(\frac{(\log(1-xt))^2}{t}\right) \leq 2\frac{1}{1-x_0t}\log\frac{1}{1-x_0t}$ which is continuous for $t \in [0,1]$. To evaluate F', integrate by parts.
Solution: a) $F(x) < \infty$ for $x \in (-\infty,1]$. b) F is derivable for $x \in (-\infty,1)$ and $F'(x) = (\log(1-x))^2/x$. c) F decreases on $(-\infty,0)$ and increases on $(0,1)$.

Problem 3.1.15** Given $a > 0$, $b > 0$, prove that

$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)\,.$$

Hints: Consider the function $f(x,t) = \frac{\cos ax - \cos bx}{x^2}\,e^{-tx}$. Then $\left|\frac{\partial}{\partial t}f(x,t)\right| \leq \frac{|\cos ax - \cos bx|}{x}e^{-t_0x} \in L^1(0,\infty)$ for $t \geq t_0 > 0$. Hence, $F(t) = \int_0^\infty f(x,t)\,dx$ is derivable on $(0,\infty)$. Even more, as $\left|\frac{\partial^2}{\partial t^2}f(x,t)\right| \leq 2e^{-t_0x} \in L^1(0,\infty)$ for $t \geq t_0 > 0$, we also have that $F(t)$ is twice derivable on $(0,\infty)$. Also, as $|f(x,t)| \leq \frac{|\cos ax - \cos bx|}{x^2} \in L^1(0,\infty)$ for $t \geq 0$, we have that F is continuous on $[0,\infty)$ and so, $F(0) = \lim_{t\to 0^+} F(t)$. To compute $F''(t)$, integrate by parts and prove that $F''(t) = \frac{t}{t^2+a^2} - \frac{t}{t^2+b^2}$. Hence, $F'(t) = \log\sqrt{\frac{t^2+a^2}{t^2+b^2}} + c_1$. By dominated convergence we have that $\lim_{t\to\infty} F'(t) = 0$ and so we deduce that $c_1 = 0$. Integrate again by parts to obtain $F(t) = t\log\sqrt{\frac{t^2+a^2}{t^2+b^2}} + a\arctan\frac{t}{a} - b\arctan\frac{t}{b} + c_2$. Finally, again by dominated convergence $\lim_{t\to\infty} F(t) = 0$ and so $c_2 = \frac{\pi}{2}(b-a)$, since $\lim_{t\to\infty} t\log\frac{t^2+a^2}{t^2+b^2} = 0$ by L'Hopital Rule.