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Departamento de Matemáticas

Integration and Measure. Problems

Chapter 3: Integrals depending on a parameter

Section 3.2: Fourier transform

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2 Integrals depending on a parameter

3.2. Fourier transform

Problem 3.2.1 Prove that if $f \in L^1(\mathbb{R})$ and $f > 0$, then $|\hat{f}(\omega)| < \hat{f}(0)$ for every $\omega \neq 0$.

Hint: The inequality $|\hat{f}(\omega)| \leq \hat{f}(0)$ is easy. If α denotes the complex argument of $\hat{f}(\omega)$, then $|\hat{f}(\omega)| = \hat{f}(\omega)\, e^{-i\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i(\omega x-\alpha)}dx$. Now, take real parts in the equality $|\hat{f}(\omega)| = \hat{f}(0)$ to conclude that, a fortiori, $\omega = 0$.

Problem 3.2.2 Given $\alpha > 0$, compute the Fourier transform of the following functions:

1) $f(x) = e^{-\alpha\|x\|}$,	2) $f(x) = \frac{2\alpha}{x^2+\alpha^2}$,
3) $f(x) = \chi_{[-\alpha,\alpha]}(x)$,	4) $f(x) = x\chi_{[-\alpha,\alpha]}(x)$,
5) $f(x) = \chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)$,	6) $f(x) = \|x\|\chi_{[-\alpha,\alpha]}(x)$,
7) $f(x) = \delta_0(x)$,	8) $f(x) = \frac{\sin\alpha x}{x}$,
9) $f(x) = (\alpha - \|x\|)\,\chi_{[-\alpha,\alpha]}$,	10) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
11) $f(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}$,	12) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
13) $f(x) = \frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}$,	14) $f(x) = \frac{1}{x}$,
15) $f(x) = \delta_{x_0} + \delta_{-x_0}$,	16) $f(x) = \delta_{x_0} - \delta_{-x_0}$,
17) $f(x) = e^{-\pi(x-3)^2}$,	18) $f(x) = e^{-i\pi(x+1)^2}$.

Problem 3.2.3 Calculate the Fourier transform of the Gaussian function $f(x) = e^{-x^2}$.

Hint: Note that the imaginary part of $\hat{f}(\omega)$ is zero. To compute the real part use the theorem on derivation of parametric integrals ($\left|\frac{\partial}{\partial\omega}\left[e^{-x^2}\cos(\omega x)\right]\right| \leq |x|e^{-x^2} \in L^1(\mathbb{R})$). Integrating by parts prove that $\frac{d}{d\omega}[\hat{f}(\omega)] = -\frac{\omega}{2}\,\hat{f}(\omega)$. Recall that $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$.

Problem 3.2.4 For $\alpha > 0$, calculate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2\alpha x}{x^2}\, dx\,.$$

Hint: Use Plancherel's theorem and part 8) of Exercise 3.2.2.

Problem 3.2.5 Find a particular solution of the equation $u'' - u = f(x)$ by taking Fourier transforms in both sides of the equation.

Problem 3.2.6 Find a solution of the initial value problem for the heat equation on $\mathbb{R}\times(0,\infty)$ by taking Fourier transforms in the x-variable in both members of the equations:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,,\\ u(x,0) \;=\; f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 3.2.7 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t) + c\,\frac{\partial}{\partial x}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,,\\ u(x,0) \;=\; f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 3.2.8 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; \frac{\partial^2}{\partial x^2}u(x,t) - 2\,\frac{\partial}{\partial x}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,,\\ u(x,0) \;=\; e^{-x^2}, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 3.2.9 Find a solution of the initial value problem for the diffusion equation with absorption:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t) - c\,u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) \;=\; f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 3.2.10 Find the solution of the initial value problem for the wave equation on $\mathbb{R} \times (0,\infty)$:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) \;=\; c^2\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) \;=\; f(x)\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t}u(x,0) \;=\; g(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 3.2.11 Prove that if f is of C^2-class (continuous with two continuous derivatives) on $\mathbb{R}$ and g is of C^1-class (continuous with one continuous derivative) on $\mathbb{R}$, then D'Alembert's formula

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\;ds\,,$$

which has been obtained in the previous problem, is effectively a solution of the initial value problem for the wave equation on $\mathbb{R} \times (0,\infty)$.

Problem 3.2.12 Find the solution of the initial value problem for the non-homogeneous wave equation on $\mathbb{R} \times \mathbb{R}$:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) \;=\; \frac{\partial^2}{\partial x^2}u(x,t) + 6\,, & \text{if } x \in \mathbb{R}\,,\ t \in \mathbb{R}\,, \\ u(x,0) \;=\; x^2\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t}u(x,0) \;=\; 4x\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

FOURIER TRANSFORMS TABLE $(x_0 \in \mathbb{R}, \alpha, \beta > 0)$

$$(TF1) \quad \mathcal{F}\big[e^{-\alpha x^2}\big](\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/(4\alpha)},$$

$$(TF2) \quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/(4\alpha)}\Big](\omega) = e^{-\alpha\omega^2},$$

$$(TF3) \quad \mathcal{F}\big[e^{-\alpha|x|}\big](\omega) = \frac{\alpha}{\pi(\omega^2+\alpha^2)}\,,$$

$$(TF4) \quad \mathcal{F}\Big[\frac{2\alpha}{x^2+\alpha^2}\Big](\omega) = e^{-\alpha|\omega|},$$

$$(TF5) \quad \mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{\sin\alpha\omega}{\pi\omega}\,, \qquad \text{if } \chi_{[a,b]}(x) = \begin{cases} 1\,, & \text{if } x \in [a,b]\,, \\ 0\,, & \text{if } x \notin [a,b]\,, \end{cases}$$

$$(TF6) \quad \mathcal{F}\Big[\frac{\sin\alpha x}{x}\Big](\omega) = \frac{1}{2}\,\chi_{[-\alpha,\alpha]}(\omega)\,,$$

$$(TF7) \quad \mathcal{F}\big[x\chi_{[-\alpha,\alpha]}(x)\big](\omega) = i\,\frac{\sin\alpha\omega - \alpha\omega\cos\alpha\omega}{\pi\omega^2}\,,$$

$$(TF8) \quad \mathcal{F}\big[\chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)\big](\omega) = i\,\frac{1-\cos\alpha\omega}{\pi\omega}\,,$$

$$(TF9) \quad \mathcal{F}\big[|x|\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{\alpha\omega\,\sin\alpha\omega + \cos\alpha\omega - 1}{\pi\omega^2}\,,$$

$$(TF10) \quad \mathcal{F}\big[(\alpha-|x|)\chi_{[-\alpha,\alpha]}(x)\big](\omega) = \frac{1-\cos\alpha\omega}{\pi\omega^2} = \frac{\sin^2(\alpha\omega/2)}{2\pi\omega^2}\,,$$

$$(TF11) \quad \mathcal{F}\big[e^{-i\alpha x^2}\big](\omega) = \frac{1}{\sqrt{4\pi\alpha}}\, e^{-i\pi/4}\, e^{i\omega^2/(4\alpha)},$$

$$(TF12) \quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}\Big](\omega) = e^{-i\alpha\omega^2},$$

$$(TF13) \quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) = e^{-\alpha|\omega|}\cos x_0\omega\,,$$

$$(TF14) \quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) = ie^{-\alpha|\omega|}\,\sin x_0\omega\,,$$

$$(TF15) \quad \mathcal{F}\Big[\frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}\Big](\omega) = \frac{1}{2\alpha\beta(\alpha^2-\beta^2)}\big(\alpha e^{-\beta|\omega|} - \beta e^{-\alpha|\omega|}\big)\,,$$

$$(TF16) \quad \mathcal{F}\Big[\frac{1}{x}\Big](\omega) = \begin{cases} -i/2\,, & \text{if } \omega < 0\,, \\ 0\,, & \text{if } \omega = 0\,, \\ i/2\,, & \text{if } \omega > 0\,, \end{cases} \qquad \text{(it's understood as the principal value)},$$

$$(TF17) \quad \mathcal{F}\big[\delta_0\big](\omega) = \frac{1}{2\pi}\,, \qquad \mathcal{F}\big[\delta_{x_0}\big](\omega) = \frac{1}{2\pi}\, e^{ix_0\omega},$$

$$(TF18) \quad \mathcal{F}\big[\delta_{x_0} + \delta_{-x_0}\big](\omega) = \frac{1}{\pi}\cos x_0\omega\,,$$

$$(TF19) \quad \mathcal{F}\big[\delta_{x_0} - \delta_{-x_0}\big](\omega) = \frac{i}{\pi}\,\sin x_0\omega\,.$$