uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 1: Measure theory.

Section 1.2: Measure spaces

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1 Measure Theory

1.2. Measure spaces

Definition 1.1 Let $(X, \mathcal{A})$ be a measurable space. A set function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ is called a (*positive*) *measure* on X if the following two conditions hold:

a) $\mu(\varnothing) = 0$.

b) μ is countably additive, i.e. if $\{A_i\}_{i=1}^{\infty}$ is a disjoint countable collection of members of $\mathcal{A}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)\,.$$

We also say that $(X, \mathcal{A}, \mu)$ is a *measure space*. A *real measure* (respectively, *complex measure*) is a set function $\mu : \mathcal{A} \longrightarrow \mathbb{R}$ (respectively, $\mu : \mathcal{A} \longrightarrow \mathbb{C}$) verifying properties a)-b).

Theorem 1.2 *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

1) *If $A_1, \ldots, A_n \in \mathcal{A}$ and are disjoint, then $\mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu(A_j)$.*

2) *If $A, B \in \mathcal{A}$, $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and, if $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.*

3) *If $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, then $\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i)$.*

4) *If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ and $\{A_n\} \subseteq \mathcal{A}$, then*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n)\,.$$

5) *If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, $\{A_n\} \subseteq \mathcal{A}$ and $\mu(A_1) < \infty$, then*

$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n)\,.$$

Example 1.3 1) Let X be any set, $\mathcal{A} = \mathcal{P}(X)$ and a function $p : X \longrightarrow [0, \infty]$. The function p is called a *weight function* and $p(x)$ the *weight* of x. If $A \subseteq X$, we define

$$\mu(A) = \sum_{x \in A} p(x) := \sup_{\{x_1, \ldots, x_n\} \subseteq A} \sum_{j=1}^{n} p(x_j)\,.$$

Then μ is a measure, the *measure defined by the weight function $p(x)$*. Two important particular cases are the following ones:

- If $p(x) = 1$ for all $x \in X$, then $\mu(A) = \operatorname{card}(A)$ and μ is called the *counting measure of X*.
- If $p(x) = 1$ for $x = a$ and $p(x) = 0$ otherwise, then $\mu(A) = 1$ if $a \in A$ and $\mu(A) = 0$ if $a \notin A$. In this case, μ is called the *δ-Dirac measure concentrated at a*.

2) Let $X = \mathbb{N}$, $A_n = \{n, n+1, n+2, \ldots\}$ and let us consider the counting measure on X. Then $\cap_n A_n = \varnothing$ but $\mu(A_n) = \infty$ for all $n \in \mathbb{N}$. This example shows that the hypothesis $\mu(A_1) < \infty$ is necessary in the last part of Theorem 1.2.

An important property of measure spaces is completeness, which is defined as follows:

Definition 1.4 Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $(X, \mathcal{A}, \mu)$ is *complete* or that μ is a *complete measure* if:

$$N \subseteq A \in \mathcal{A},\ \mu(A) = 0 \implies N \in \mathcal{A} \quad \text{and} \quad \mu(N) = 0\,.$$

The sets N are called the *null sets* for the measure μ.

Theorem 1.5 (Completion of a measure space) *Let $(X, \mathcal{A}, \mu)$ be a measure space and let*

$$\mathcal{N} := \{N \subseteq X :\ N \subseteq B \in \mathcal{A},\ \mu(B) = 0\}$$

Then

1) $\bar{\mathcal{A}} := \{A \cup N :\ A \in \mathcal{A},\ N \in \mathcal{N}\}$ *is a σ-algebra.*

2) $\bar{\mu} : \bar{\mathcal{A}} \longrightarrow [0, \infty]$ *given by $\bar{\mu}(A \cup N) = \mu(A)$ is well-defined and is a measure (extending μ).*

3) $(X, \bar{\mathcal{A}}, \bar{\mu})$ *is complete and is called the completion of $(X, \mathcal{A}, \mu)$.*

Definition 1.6 (True almost everywhere properties).
We say that a property is *true almost everywhere* if the set of points where the property does not hold has zero measure. We shall use the notation a.e. for "almost everywhere".

Remark 1.7 To allow the use of ∞ in Measure Theory the following conventions are used:
1) $a + \infty = \infty + a = \infty$ for $0 \leq a \leq \infty$.
2) $a \cdot \infty = \infty \cdot a = \infty$ for $0 < a \leq \infty$, but $0 \cdot \infty = \infty \cdot 0 = 0$!!!
3) Cancelation laws:

$$a + b = a + c \implies b = c \qquad \text{only if} \quad -\infty < a < \infty\,,$$

$$ab = ac \implies b = c \qquad \text{only if} \quad 0 < a < \infty\,.$$