uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Integration and Measure

Chapter 1: Measure theory.

Section 1.3: Construction of measures

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

1 Measure Theory

1.3. Construction of measures

1.3.1. Outer measures

Definition 1.1 An <u>outer measure</u> is a set function $\mu^* : \mathcal{P}(X) \longrightarrow [0, \infty]$ verifying:

(1) $\mu^*(\varnothing) = 0$.

(2) $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

(3) If $\{A_j\}$ is a collection of subsets of X then: $\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

We are trying to find a collection $\mathcal{M} \subseteq \mathcal{P}(\mathcal{X})$ such that $\mu^*|_{\mathcal{M}}$ is a measure. The following theorem gives an original solution:

Theorem 1.2 (Caratheodory's theorem)
Let μ^* be an outer measure on X. A subset $M \subseteq X$ is said to be <u>μ^*-measurable</u> if

$$\mu^*(A) = \mu^*(A \cap M) + \mu^*(A \setminus M), \qquad \forall A \subseteq X.$$

Let $\mathcal{M} = \{M \subseteq X : \ M \text{ is } \mu^*\text{-measurable}\}$. Then

i) $\mathcal{M}$ is a σ-algebra.

ii) $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure.

Hence, in order to construct measures we must construct outer measures: a simpler problem!!!

Definition 1.3 A collection $\mathcal{E} \subseteq \mathcal{P}(\mathcal{X})$ is said to be a <u>semi-algebra</u> if

(1) $\varnothing \in \mathcal{E}$.

(2) $E, F \in \mathcal{E} \implies E \cap F \in \mathcal{E}$.

(3) $E \in \mathcal{E} \implies E^c = F_1 \cup F_2 \cup \cdots \cup F_n$ where $F_j \in \mathcal{E}$ and are disjoint.

Example 1.4 If $X = \mathbb{R}$, then the family $\mathcal{E}$ of semi-open intervals of type $[a, b)$, $[a, \infty)$, $(-\infty, b)$ and $(-\infty, \infty)$ is a semi-algebra.

Definition 1.5 Let $\mathcal{E}$ be a semi-algebra. A set function $\mu_0 : \mathcal{E} \longrightarrow [0, \infty)$ is said to be <u>countably additive</u> if

$$\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{E} \quad \text{disjoint}, \quad \bigcup_{j=1}^{\infty} E_j \in \mathcal{E} \implies \mu_0\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \mu_0(E_j).$$

Theorem 1.6 (Caratheodory-Hopf's extension theorem)
Let $\mathcal{E} \subseteq \mathcal{P}(\mathcal{X})$ be a semi-algebra and $\mu_0 : \mathcal{E} \longrightarrow [0, \infty)$ be a countably additive function. Let us define for all $A \in \mathcal{P}(X)$:

$$\mu^*(A) = \inf\Big\{ \sum_{j=1}^{\infty} \mu_0(E_j) : \ E_j \in \mathcal{E}, A \subseteq \bigcup_{j=1}^{\infty} E_j \Big\}.$$

Then:

1) μ^* is an outer measure and $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure which is an extension of μ_0, i.e. $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{E}$.

2) If μ_0 is σ-finite (i.e. if $X = \cup_{j=1}^{\infty} X_j$ with $\mu_0(X_j) < \infty$), then

2.a) *μ is the unique measure which is an extension of μ_0 to $\sigma(\mathcal{E})$.*

2.b) *$\mathcal{M} = \overline{\sigma(\mathcal{E})}$, i.e. $\mathcal{M}$ is the completion relative to μ of $\sigma(\mathcal{E})$.*

1.3.2. Lebesgue measure

A *semi-open interval in* $\mathbb{R}$ is an interval of one of the types $[a,b)$, $[a,\infty)$, $(-\infty,b)$ or $(-\infty,\infty)$. A *semi-open interval in* $\mathbb{R}^n$ is a set of the type $I = I_1 \times \cdots \times I_n$, where each I_j is a semi-open interval in $\mathbb{R}$.

Proposition 1.7 *The collection $\mathcal{E}$ of all semi-open intervals in $\mathbb{R}^n$ is a semi-algebra and $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R}^n)$.*

Definition 1.8 Let $I = I_1 \times \cdots \times I_n$ be a semi-open interval in $\mathbb{R}^n$. We define the *elemental measure* of I as:

$$\mu_0(I) = (b_1 - a_1)\cdots(b_n - a_n)\,, \qquad \text{where } I_j = [a_j, b_j)\,.$$

If some I_j is not bounded then we define $\mu_0(I) = \infty$ and if $I_j = \varnothing$ for some j, then $\mu_0(I) = 0$.

It can be proved that μ_0 is countably additive and σ-finite. Hence, by Caratheodory-Hopf's extension theorem

Theorem 1.9 *There exists a unique measure space $(\mathbb{R}^n, \mathcal{M}, m)$ such that $\mathcal{M} = \overline{\mathcal{B}(\mathbb{R}^n)}$ and $m|_{\mathcal{E}} = \mu_0$. In particular,*

i) For all $M \in \mathcal{M}$, $M = B \cup N$ where $B \in \mathcal{B}(\mathbb{R}^n)$ and $m(N) = 0$.

ii) For all $N \in \mathcal{M}$ with $m(N) = 0$ there exists $A \in \mathcal{B}(\mathbb{R}^n)$ with $N \subseteq A$ and $\mu(A) = 0$.

This unique measure m is called the *Lebesgue measure* on $\mathbb{R}^n$.

Remarks:
1) $\mathcal{N} = \{N \in \mathcal{M} : \; m(N) = 0\}$.
2) $\{N_j\}_{j=1}^{\infty} \subseteq \mathcal{N} \implies \bigcup_{j=1}^{\infty} N_j \in \mathcal{N}$.
3) $a \in \mathbb{R}^n \implies \{a\} \in \mathcal{N}$
4) $A \subseteq \mathbb{R}^n$ countable $\implies A \in \mathcal{N}$.
5) There exist non-countable sets in $\mathcal{N}$. For example, the Cantor ternary set.
6) If $H \subseteq \mathbb{R}^n$ is a translated $(n-1)$-dimensional hyperplane, then $m(H) = 0$.
7) $\mathcal{B}(\mathbb{R}^n) \subsetneq \mathcal{M} \subsetneq \mathcal{P}(\mathbb{R}^n)$.
8) If $A \subset \mathbb{R}^n$ is an open set, then $m(A) > 0$.
9) If $K \subset \mathbb{R}^n$ is a compact set, then $m(K) < \infty$.
10) m is regular, i.e. for all $A \in \mathcal{M}$,

$$m(A) = \inf\{m(V) : \; A \subset V \text{ open}\} = \sup\{m(K) : \; K \subset A \text{ compact}\}.$$

If X is a topological space with its Borel σ-algebra, we say that a measure μ on X is a *Radon measure* if every compact set has finite μ-measure.

Theorem 1.10 *Lebesgue measure is the unique (up to multiplicative constants) translations-invariant Radon measure on $\mathbb{R}^n$:*

i) $(\mathbb{R}^n, \mathcal{M}, m)$ is translations-invariant, i.e.

$$A \in \mathcal{M},\; a \in \mathbb{R}^n \implies a + A \in \mathcal{M} \text{ and } m(a + A) = m(A)\,.$$

ii) If $\mu : \mathcal{M} \longrightarrow [0,\infty]$ is a Radon measure which is translations-invariant, then $\mu = k\,m$ for some positive constant k.

1.3.3. Borel-Stieltjes measures on $\mathbb{R}$

What are the Radon measures on $\mathcal{B}(\mathbb{R})$? Let us observe that the function $g(t) := \mu\big((-\infty, t)\big)$ is increasing and $\mu([a,b)) = g(b) - g(a)$. These equations allow to determine g from μ and vice versa.

Theorem 1.11 *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function. Then, there exists a unique Radon measure $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$ such that $\mu([a,b)) = g(b^-) - g(a^-)$. This measure $\mu = \mu_g$ is called the Borel-Stieltjes measure with distribution function g.*

Here $g(x_0^-) = \lim_{x\to x_0^-} g(x)$.

Remark 1.12 An increasing function $g : \mathbb{R} \longrightarrow \mathbb{R}$ only can be discontinuous on a countable set. By defining $\tilde{g}(t) = g(t^-)$ we obtain an increasing function which also is left-continuous and as

$$\mu_g([a,b)) = g(b^-) - g(a^-) = \tilde{g}(b) - \tilde{g}(a) = \mu_{\tilde{g}}([a,b))$$

applying Caratheodory-Hopf's theorem we deduce that $\mu_g = \mu_{\tilde{g}}$.

Theorem 1.13 *If $\mu : \mathcal{B}(\mathbb{R}) \longrightarrow [0,\infty]$ is a Radon measure, then there exists an increasing and left-continuous function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mu = \mu_g$. Besides, g is unique up to add constants.*

Example 1.14 1) If $g(t) = t$ then $\mu_g = m$ with m the Lebesgue measure on $\mathbb{R}$.
2) If $g = \chi_{(0,\infty)}$ then $\mu_g = \delta_0$, the δ-Dirac measure concentrated at $x = 0$.

1.3.4. Image measure

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\Phi : X \longrightarrow Y$ be a mapping. We define:

$$\mathcal{B} = \Phi(\mathcal{A}) := \{B \subseteq Y : \ \Phi^{-1}(B) \in \mathcal{A}\},$$

and the set function: $\nu = \Phi(\mu) : \mathcal{B} \longrightarrow [0,\infty]$ given by $\nu(B) = \mu(\Phi^{-1}(B))$ for $B \in \mathcal{B}$.

Theorem 1.15 *$(Y, \mathcal{B}, \nu)$ is a measure space. It is complete if $(X, \mathcal{A}, \mu)$ is. It is called the <u>image measure space of $(X, \mathcal{A}, \mu)$</u>.*

Examples

1) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and strictly increasing. Therefore g is injective and it has continuous inverse g^{-1} (in fact, g is a homeomorphism). Then $g^{-1}(m) = \mu_g$, i.e. the image measure by g^{-1} of Lebesgue measure coincides with the Borel-Stieltjes measure with distribution function g.

2) $g : (0,\infty) \longrightarrow \mathbb{R}$ given by $g(t) = \log t$. Then $\mu_g = g^{-1}(m) = e^m$ is rotations-invariant.

3) The $(n-1)$-dimensional Lebesgue measure σ on the sphere $S_{n-1} = \{x \in \mathbb{R}^n : \ \|x\| = 1\}$ of $\mathbb{R}^n$ is defined as a image measure on the following way: let us consider the projection mapping $\pi : B_n \setminus \{0\} \longrightarrow S_{n-1}$ given by $\pi(x) = x/\|x\|$. Here $B_n = \{x \in \mathbb{R}^n : \ \|x\| < 1\}$ is the n-dimensional open ball. Then we define: $\sigma = n \cdot \pi(m)$ where m denotes the Lebesgue measure on B_n. Hence

$$\sigma(U) = n\, m(\pi^{-1}(U)), \qquad \forall U \in \mathcal{B}(S_{n-1}).$$

This measure σ is rotations-invariant.

1.3.5. Product measure

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.

Definition 1.16 Let $X \times Y := \{(x,y) : \ x \in X, y \in Y\}$. The *<u>product σ-algebra</u>* is defined as

$$\mathcal{A} \otimes \mathcal{B} := \sigma\big(\mathcal{E}\big), \qquad \mathcal{E} = \{A \times B : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Theorem 1.17 *There exists a unique measure $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \longrightarrow [0,\infty]$ such that*

$$(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B), \qquad \forall A \in \mathcal{A}, \forall B \in \mathcal{B}.$$

Definition 1.18 The <u>product measure space</u> of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ is $(X \times Y, \overline{\mathcal{A} \otimes \mathcal{B}}, \mu \otimes \nu)$.

Proposition 1.19 *The product measure space of the Lebesgue measure spaces on $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ is the Lebesgue measure space on $\mathbb{R}^n$.*